Quilts of Alternating Sign Matrices

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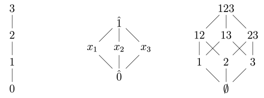
Posets

Let P and Q be finite ranked posets with least and greatest elements. Important examples of such posets are:

- C_n for $n \geq 0$ is the **chain** of rank n;
- $A_2(j)$ for $j \geq 1$ is the **antichain** of j elements, with $\hat{0}$, $\hat{1}$;
- B_n for $n \geq 1$ is the **Boolean lattice** of rank n.



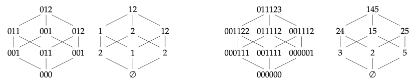
The main definition

A **quilt of alternating sign matrices of type (P,Q)** is a map $f\colon P \times Q \longrightarrow \mathbb{N}$ satisfying

- $f(x,y) = 0$ if $x = \hat{0}_P$ or $y = \hat{0}_Q$,
- $f(\hat{1}_P, \hat{1}_Q) = \min\{\operatorname{rank} P,\ \operatorname{rank} Q\}$, and
- if $(x,y) \lessdot (x',y')$ in $P \times Q$, then $f(x',y') - f(x,y) \in \{0,1\}$. (**Boolean growth**).

We will also call such a map an **ASM quilt** or just a **quilt** for short. The set of all quilts of type (P,Q) will be denoted by $\operatorname{Quilts}(P,Q)$. The set $\operatorname{Quilts}(P,Q)$ is partially ordered by entrywise comparison.

Examples (also in MT form):



Type (B_3, C_2) Type (B_3, C_5)

Motivation: CSMs, ASMs, and Fubini-Bruhat order

Take a real matrix A of size $k \times n$ with rank $\min\{k,n\}$. For $0 \leq i \leq k$, $0 \leq j \leq n$, let $f(i,j)$ denote the **rank of the submatrix** of A consisting of the i bottommost rows and j leftmost columns. The resulting matrix is a map $f\colon C_k \times C_n \to \mathbb{N}$ satisfying the quilt properties. We take the above to be the definition of a **corner sum matrix (CSM)**.

For a CSM f, take $a_{i,j} = f(i,j) - f(i,j-1) - f(i-1,j) + f(i-1,j-1)$. Now all the entries are 0, 1, or -1, in each row and each column the non-zero entries alternate, the leftmost non-zero entry in every row and the bottommost non-zero entry in every column is 1, if $k \leq n$, the rightmost non-zero entry in every row is 1, and if $k \geq n$, the topmost non-zero entry in every column is 1. This is a **rectangular ASM**.

If we replace the chains above by posets P and Q, we get a quilt. We can think of quilts as encoding **collections of alternating sign matrices**, one for each pair of maximal chains in the two posets, appropriately "pieced" together.

Quilt lattices include lattices on matroids, flag matroids, Bruhat order on permutations, and the medium roast Fubini-Bruhat order on the Pawlowski-Rhoades varieties from the geometry of spanning line configurations, which was our original motivation.

The quilt lattice

Theorem

The poset $\operatorname{Quilts}(P,Q)$ is a distributive lattice ranked by

$$\operatorname{quiltrank} f = \sum_{x \in P,\, y \in Q} f(x,y) - \sum_{x \in P,\, y \in Q} f_{\hat{0}}(x,y),$$

where $f_{\hat{0}}(x,y) = \max\{0, \operatorname{rank} x + \operatorname{rank} y - \max\{n,k\}\}$ is the least element of $\operatorname{Quilts}(P,Q)$. The greatest element of $\operatorname{Quilts}(P,Q)$ is $f_{\hat{1}}(x,y) = \min\{\operatorname{rank} x,\ \operatorname{rank} y\}$.

Theorem

- If φ is an (involutive) antiautomorphism of P and $\operatorname{rank} P \geq \operatorname{rank} Q$, then
  $$\Phi\colon\ \operatorname{Quilts}(P,Q) \to \operatorname{Quilts}(P,Q),\ \text{ where } \Phi f(x,y) = \operatorname{rank} y - f(\varphi(x), y)$$
  is an (involutive) antiautomorphism of the lattice $\operatorname{Quilts}(P,Q)$.
- Given an involutive antiautomorphism $\varphi\colon P \to P$, $\operatorname{rank} P \geq 2$, there is a faithful action of the dihedral group D_4 acting on $\operatorname{Quilts}(P,P)$.

Enumerative properties

A very famous theorem states that $|\operatorname{Quilts}(C_n, C_n)| = |\operatorname{ASM}_n| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. It turns out that computing $|\operatorname{Quilts}(P,Q)|$ for general P and Q is a #P-complete problem, so we cannot generalize this to arbitrary P and Q, but we have identified some special cases with elegant enumerative formulas.

Antichain quilts

An **antichain quilt** is a quilt of type $(P, A_2(j))$ for some poset P and some j. A set $S \subseteq P$ is **convex** if $x, y \in S$ implies $[x,y] \subseteq S$. We say that S is a **cut set** if it intersects every maximal chain in P. The **number of antichains** in S is $\alpha(S)$.

Theorem

We have $|\operatorname{Quilts}(P, A_2(j))| = \sum_C \alpha(C)^j$, where the sum is over all subsets C of $P \setminus \{\hat{0}_P, \hat{1}_P\}$ that are convex cut sets of P. In particular, as j goes to infinity, we have $|\operatorname{Quilts}(P, A_2(j))| \sim \alpha(P \setminus \{\hat{0}_P, \hat{1}_P\})^j$.

Chain quilts

A **chain quilt** is a quilt of type (P, C_n) for some poset P and some n.

Theorem

For a fixed poset P, the number of chain quilts of type (P, C_n), $n \geq \operatorname{rank} P$, is given by a polynomial in n, namely $|\operatorname{Quilts}(P, C_n)| = \sum_{m=k}^{b(P)} |F_m(P)| \binom{n}{m}$, where $b(P) = \sum_{x \in P} \operatorname{rank} x$ and $F_m(P)$ is the set of **m-fundamental quilts**. Also, $|\operatorname{Quilts}(P, C_n)| \sim \frac{|S(P)|}{b(P)!} \cdot n^{b(P)}$, where $S(P)$ is the set of **standard quilts**.

As an application, fix k. Then the number of rectangular ASMs of size $k \times n$, $k \leq n$, is a polynomial in n of degree $\binom{k+1}{2}$ with leading coefficient $\frac{\prod_{i=0}^{k-1}(2i)!}{\prod_{i=0}^{k-1}(k+i)!}$.

Boolean quilts

A **Boolean quilt** is a quilt of type (P, B_n) for some poset P and some n.

Theorem

For a fixed poset P, there exist positive numbers A_P and B_P so that if $n \geq \operatorname{rank} P$, we have

$$A_P^{\binom{n}{\lfloor n/2 \rfloor}} \leq |\operatorname{Quilts}(P, B_n)| \leq B_P^{\binom{n}{\lfloor n/2 \rfloor}}.$$

We have

$$2^{\binom{k}{\lfloor k/2 \rfloor}\binom{n}{\lfloor n/2 \rfloor}} \leq |\operatorname{Quilts}(B_k, B_n)| \leq 2^{k 2^{k-1}(1 + c \ln n/\sqrt{n})\binom{n}{\lfloor n/2 \rfloor}}$$

for $n \geq 2k$ and some constant $c > 0$.

Open questions

1. **Representability.** Characterize the quilts of type (B_k, B_n) such that there exists a matrix $A \in \mathbb{R}^{k \times n}$, $\operatorname{rank} A = \min\{k,n\}$, and $f(I,J)$ is equal to the rank of the matrix obtained by taking rows in I and columns in J in the matrix A.
2. **Dynamical properties.** Study toggling, rowmotion, orbits etc.
3. **Quilt lattice.** Study further properties, like irreducible elements.
4. **Generalizing ASM.** The literature on alternating sign matrices provide a rich source of problems for quilts. What are the inversions and descents for quilts following Hamaker–Reiner? Is there an analog of ASM varieties following Weigandt? Is there a Hopf algebra interpretation for quilts a la Cheballah–Giraudo–Maurice?