Standard Monomials for Positroid Varieties

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Introduction

Classical work of Hodge [1] described a particular set of bases for the homogeneous coordinate rings of the Grassmannian $\mathrm{Gr}(k,n)$ and its Schubert varieties under the Plücker embedding. Building on Hodge's ideas, Seshadri initiated the study of standard monomial theory (SMT), with the aim of giving standard monomial bases for the space of global sections of line bundles over a flag variety G/P.
The main goal of our work is to extend the work of Hodge to **positroid varieties** in $\mathrm{Gr}(k,n)$, based on works of Knutson–Lam–Speyer. We prove:

- an explicit description of standard monomials for positroid varieties based on semistandard Young tableaux.
- an explicit Gröbner basis for positroid varieties with respect to the Hodge degeneration.
- a connection between the *promotion* (resp. *evacuation*) on rectangular semistandard Young tableaux and rotations (resp. reflections) of positroid varieties.
- a formula for characters of *cyclic Demazure modules*, resolving a problem of Lam.

Plücker Coordinates

For $k \le n \in \mathbb{Z}_{>0}$, the **Grassmannian** $\mathrm{Gr}(k,n)$ is

$$\mathrm{Gr}(k,n) = \{W \subseteq \mathbb{C}^n : \dim(W) = k\}.$$

We can represent $V \in \mathrm{Gr}(k,n)$ by a full rank $k \times n$ matrix

$$\widetilde{V} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} — w_1 — \\ \vdots \\ — w_k — \end{bmatrix}$$

where V is the row span of $\widetilde{V}$. The **Plücker embedding** is the map $\mathrm{Gr}(k,n) \to \mathbb{P}(\Lambda^k(\mathbb{C}^n))$ sending V to $[w_1 \wedge \cdots \wedge w_k]$.
Set $R(k,n) := \mathbb{C}[[\mathbf{a}] : \mathbf{a} \in \binom{[n]}{k}]$ to be the homogeneous coordinate ring of $\mathbb{P}(\Lambda^k(\mathbb{C}^n))$. We use the convention that for any permutation $\sigma \in S_n$,

$$[a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(k)}] = (-1)^{\ell(\sigma)} \cdot [a_1, \ldots, a_k].$$

We will often represent $\mathbf{m} = \prod_{i=1}^d [\mathbf{a}^{(i)}] \in R(k,n)$ as a $k \times d$ tableau, where column i is strictly increasing with entries in $\mathbf{a}^{(i)}$ for all i.

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 6 & 5 \\ \hline\end{array} \iff [1,2,6] \cdot [3,4,5]$$

Denote $B(k,n,d)$ the set of semistandard tableaux of shape $k \times d$ with entries in $[n]$.
The homogeneous coordinate ring of $\mathrm{Gr}(k,n)$ is $\mathbb{C}[\mathrm{Gr}(k,n)] = R(k,n)/\mathcal{J}$. It is the direct sum of its graded pieces:

$$\mathbb{C}[\mathrm{Gr}(k,n)] = \bigoplus_{d=0}^{\infty} \mathbb{C}[\mathrm{Gr}(k,n)]_d,$$

Let $<_\omega$ be a degree revlex monomial order on $R(k,n)$. For a subvariety $X \subseteq \mathrm{Gr}(k,n)$, let $\mathcal{J}_X$ be its defining ideal under the Plücker embedding. A monomial $\mathbf{m} = \prod_{i=1}^d [\mathbf{a}^{(i)}]$ is called a **standard monomial** for X if $\mathbf{m} \notin \mathrm{init}_{<_\omega}(\mathcal{J}_X)$.

Theorem [1]

The monomials $\{\mathbf{m} \in B(k,n,d)\}$ are the degree d standard monomials for $\mathrm{Gr}(k,n)$. They form a basis of $\mathbb{C}[\mathrm{Gr}(k,n)]_d$.

Positroid Varieties

For $1 \le i,j \le n$, define the cyclic interval

$$[i,j]^\circ = \begin{cases} [i,j] & \text{if } i \le j \\ [n] \setminus [j+1, i-1] & \text{if } i > j \end{cases}.$$

The positroid varieties Π_f are a special family of subvarieties of $\mathrm{Gr}(k,n)$.

- they contain Schubert varieties as special cases
- rotation/reflection of positroid varieties are positroid varieties
- each positroid variety Π_f can be written as:

$$\Pi_f = \bigcap_{(i,j) \in \mathrm{ess}(f)} \Pi_{[i,j]^\circ \le r(f)_{i,j}} \qquad (1)$$

where
$\Pi_{[i,j]^\circ \le r} = \{V \in \mathrm{Gr}(k,n) : \mathrm{rank}(\widetilde{V}_{[i,j]^\circ}) \le r\}$.

Let $\mathcal{J}_f \subset R(k,n)$ be the defining ideal of Π_f.

$$\mathbb{C}[\Pi_f] = R(k,n)/\mathcal{J}_f = \bigoplus_{d=0}^{\infty} \mathbb{C}[\Pi_f]_d.$$

Standard Monomials

For a cyclic interval $S = [i,j]^\circ$ and $r < |[i,j]^\circ|$, set

- $S^\vee := [j+1, i-1]^\circ$.
- $r^\vee := n - k - |[i,j]^\circ| + r$
- For $\mathbf{m} = \prod_{i=1}^d [\mathbf{a}^{(i)}] \in R(k,n)$, define $\mathbf{m}^\vee$ to be $\prod_{i=1}^d [\mathbf{a}^{(i)\vee}] \in R(n-k,n)$, where $\mathbf{a}^{(i)\vee} := [n] \setminus \mathbf{a}^{(i)}$ for all $i \in [d]$.

For a SSYT, a **generalized antidiagonal** for the rank condition $[i,j] \le r$ is a vertical strip of size $r+1$ with entries in $[i,j]$ that are strictly increasing from NE to SW.

Example

Let $n = 5$, $k = 3$, and consider the positroid variety $\Pi_{[2,4]\le 2}$. The following $\mathbf{m} \in B(k,n,d)$ contains a generalized antidiagonal for the rank condition $[2,4] \le 2$:

$$\begin{array}{|c|}\hline \mathbf{2} \\ \hline \mathbf{3} \\ \hline \mathbf{4} \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 1 & \mathbf{2} \\ \hline 2 & \mathbf{3} \\ \hline \mathbf{4} & 5 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 1 & \mathbf{2} \\ \hline 3 & \mathbf{3} \\ \hline \mathbf{4} & 5 \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline 1 & \mathbf{2} \\ \hline \mathbf{3} & 4 \\ \hline \mathbf{4} & 5 \\ \hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 1 & \mathbf{2} \\ \hline 2 & \mathbf{3} & 4 \\ \hline \mathbf{4} & 5 & 5 \\ \hline\end{array}.$$

For $S = [i,j]^\circ$, define $B_{S \le r}(n,k,d)$ to be the set of monomials $\mathbf{m}$ corresponding to SSYT of shape $k \times d$ with entries in $[n]$ such that

- $\mathbf{m}$ does not have any generalized antidiagonal for $S \le r$ if $i \le j$;
- $\mathbf{m}^\vee$ does not have any generalized antidiagonal for $S^\vee \le r^\vee$ if $i > j$;

Define

$$B_f(k,n,d) = \bigcap_{(i,j) \in \mathrm{ess}(f)} B_{[i,j]^\circ \le r(f)_{i,j}}(k,n,d).$$

Theorem [2]

The monomials $\{\mathbf{m} \in B_f(k,n,d)\}$ are the degree d standard monomials for Π_f. They form a basis of $\mathbb{C}[\Pi_f]_d$.

Promotion

Let **promotion** be the map $\mathrm{prom} : B(k,n,d) \longrightarrow B(k,n,d)$ defined as follows:

- If $\mathbf{m} \in B(k,n,d)$ does not contain n, then increase each entry of $\mathbf{m}$ by 1.
- If $\mathbf{m}$ contains n, replace each n with $\bullet$ and perform the **jeu de taquin (jdt)** slide:

$$\begin{array}{|c|c|}\hline a & c \\ \hline b & \bullet \\ \hline\end{array} \longrightarrow \begin{cases} \begin{array}{|c|c|}\hline a & \bullet \\ \hline b & c \\ \hline\end{array} & \text{if } b \le c \text{ or } a,b \text{ do not exist,} \\ \begin{array}{|c|c|}\hline a & c \\ \hline \bullet & b \\ \hline\end{array} & \text{if } b > c \text{ or } a,c \text{ do not exist} \end{cases}$$

until no longer possible. Replace each $\bullet$ with 0 and increase all entries by 1.

Let **evacuation** be the map $\mathrm{evac} : B(k,n,d) \to B(k,n,d)$ defined by:

- first replace every entry x with $n+1-x$,
- then rotate the tableau by 180°.

Theorem [2]

Promotion (resp. evacuation) bijects the set of standard monomials for a positroid variety Π_f and those of its cyclic shift $\chi(\Pi_f)$ (resp. its reflection $\Pi_{f^*} = w_0 \cdot \Pi_f$).

Example

Consider the positroid variety $\Pi_{[2,4]\le 1} \subset \mathrm{Gr}(3,7)$, then $\chi(\Pi_f) = \Pi_{[3,5]\le 2}$ and $w_0 \cdot \Pi_f = \Pi_{[4,6]\le 2}$.

$$\begin{array}{|c|c|}\hline 1 & \mathbf{2} \\ \hline \mathbf{3} & 5 \\ \hline \mathbf{4} & 7 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \bullet \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \bullet \\ \hline 4 & 5 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bullet & 3 \\ \hline 4 & 5 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 1 & 3 \\ \hline 4 & 5 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & \mathbf{3} \\ \hline 2 & \mathbf{4} \\ \hline \mathbf{5} & 6 \\ \hline\end{array}.$$

$$\mathrm{evac} : \begin{array}{|c|c|}\hline 1 & \mathbf{2} \\ \hline \mathbf{3} & 5 \\ \hline \mathbf{4} & 7 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 7 & 6 \\ \hline 5 & 3 \\ \hline 4 & 1 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 1 & \mathbf{4} \\ \hline 3 & \mathbf{5} \\ \hline \mathbf{6} & 7 \\ \hline\end{array}$$

References

[1] W. V. D. Hodge.
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