

# A new definition for *m*-Cambrian lattices

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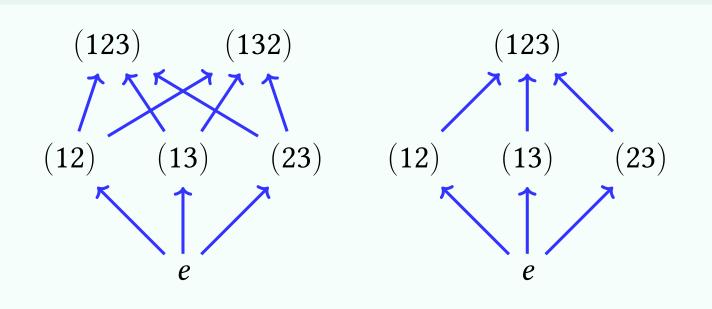
The Cambrian lattices, introduced by Reading [Reading, 2006], generalize the Tamari lattice [Tamari, 1962] to any choice of Coxeter element in any finite Coxeter group. They are further generalized to the *m*-Cambrian lattices [Stump et al., 2020]. We propose a new definition for *m*-Cambrian lattices, where the objects are *m*-multichains in the noncrossing partition lattice, with an explicit comparison criterion.

## Noncrossing partitions and Cambrian lattice

#### Definition

A (standard) Coxeter element c in a Coxeter group W is a product of all simple reflections in some order. Coxeter elements are always maximal in the absolute order Abs(W).

The **noncrossing partition lattice** NCL(W, c) is the interval [e, c] in the absolute order Abs(W).

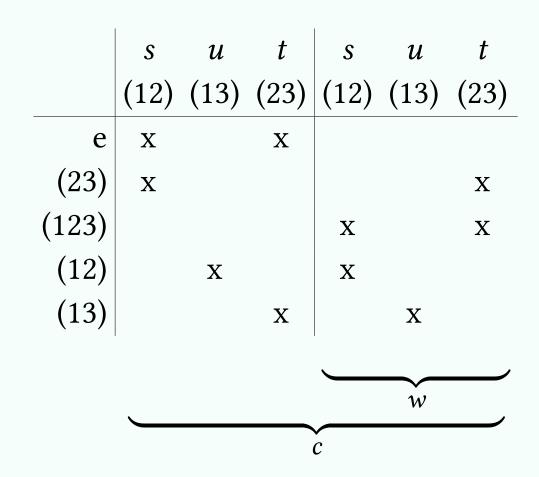


#### Example

The absolute order on  $\mathfrak{S}_3$  (left) and the noncrossing partition lattice  $NCL(\mathfrak{S}_3,(123))$  (right).

#### Proposition [C. Athanasiadis, T. Brady, and C. Watt, '07]

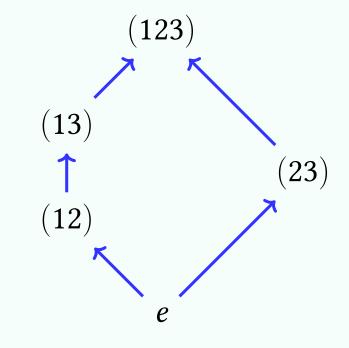
The choice of a Coxeter element c leads to a total order R(c) on all reflections. Each c-noncrossing partition has a unique  $\mathcal{R}$ -word with letters increasing in R(c). They are thus in bijection with subwords of the word  $R(c)^2$  that are reduced  $\mathcal{R}$ -words for c, which are called **1-factorizations of** c.



#### Definition

A **rotation** of a 1-factorization of c consists in moving a cross in the first copy of R(c) to the second, conjugating every cross in between.

The Cambrian lattice Camb(W, c) is the poset on 1-factorizations obtained as the transitive closure of **rotations**.



#### Example

The Cambrian lattice  $Camb(\mathfrak{S}_3, (123))$ .

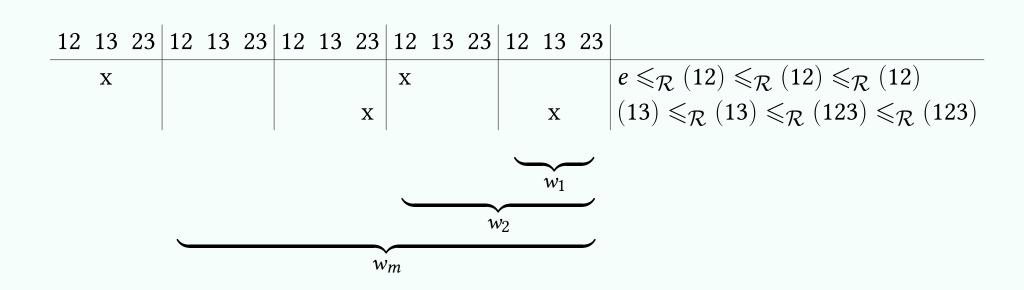
## New definition of the *m*-Cambrian lattices

#### Definition

An *m*-factorization of a Coxeter element c is a subword of  $R(c)^{m+1}$  that is a reduced  $\mathcal{R}$ -word for c.

#### Proposition [D. Armstrong, '09]

Multichains with m elements in the poset NCL(W, c) are in bijection with m-factorizations of c.



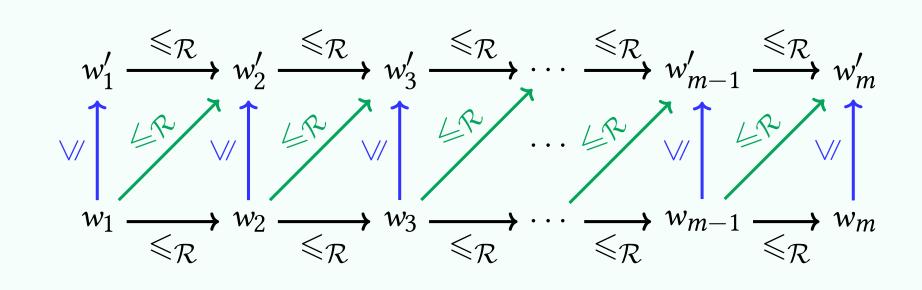
### Example

Two 4-noncrossing partitions in  $NCL(\mathfrak{S}_3, (123))$ .

#### Definition

For *m*-noncrossing partitions  $w_{(m)}, w'_{(m)}$ , we set  $w_{(m)} \leq_{(m)} w'_{(m)}$  if

- 1. Vertical condition: For all  $1 \leqslant i \leqslant m$ ,  $w_i \leqslant w'_i$  in Camb(W, c);
- 2. **Diagonal condition**: For all  $1 \le i < m$ ,  $w_i \le_{\mathcal{R}} w'_{i+1}$  in NCL(W, c).



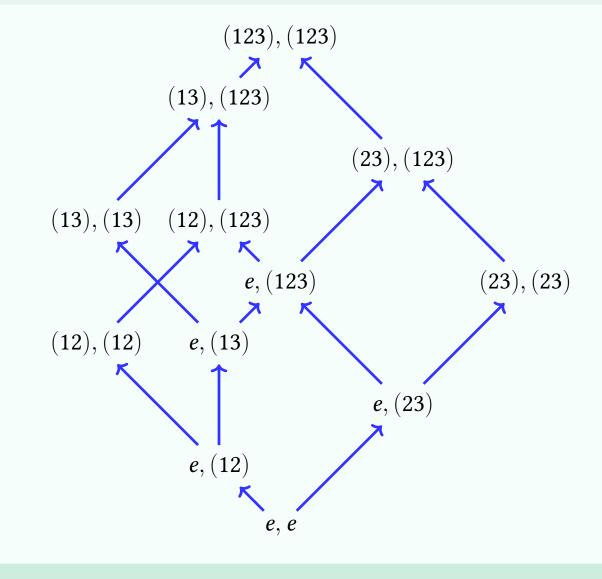
## Example

Since  $e \leqslant (13)$ ,  $(12) \leqslant (13)$ ,  $(12) \leqslant (123)$ , and  $(12) \leqslant (123)$  in  $\operatorname{Camb}(\mathfrak{S}_3, (123))$ , the vertical condition is satisfied.

Since  $e \leqslant_{\mathcal{R}} (12)$ ,  $(13) \leqslant_{\mathcal{R}} (13)$ , and  $(13) \leqslant_{\mathcal{R}} (123)$  in NCL( $\mathfrak{S}_3$ , (123)), the diagonal condition is also satisfied.

#### Theorem [C., Fang, Henriet, '24+]

The binary relation  $\leq_{(m)}$  is a partial order on m-noncrossing partitions, which is isomorphic to the m-Cambrian lattice  $\operatorname{Camb}^{(m)}(W,c)$ .



### Example

The 2-Cambrian lattice  $Camb^{(2)}(\mathfrak{S}_3, (123))$ .

# c-increasing chains and greedy algorithm

## Definition

If  $w \lessdot w'$  is a covering relation, define the **flip reflection** r(w, w') as the selected letter of w that is sent to the next copy.

A saturated chain  $w_0 < w_1 < ... < w_m$  is a *c*-increasing chain if its sequence  $(r(w_i, w_{i+1}))$  of flip reflections is increasing with respect to  $R(c)^{m+1}$ .

### Proposition: Unicity of *c*-increasing chain

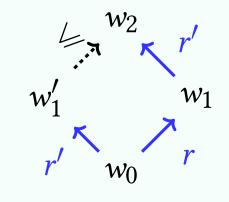
For  $w \le w'$  in the m-Cambrian lattice, if there exists a c-increasing chain from w to w', then the smallest letter of w that is not a letter of w' is the smallest flip reflection in any saturated chain from w to w'. Thus, a c-increasing chain from w to w' is **unique**.

## Proposition: Existence

If  $w \le w'$  in the m-Cambrian lattice, there **exists** a c-increasing chain from w to w'.

### Local reordering lemma

Let  $w_0 \lessdot w_1 \lessdot w_2$  in  $\operatorname{Camb}^{(m)}(W, c)$  with  $r(w_0, w_1) > r(w_1, w_2) = r'$ . Then  $r' \in w_0$ , and for  $w'_1$  the upper cover of  $w_0$  with  $r(w_0, w'_1) = r'$ , we have  $w'_1 \leqslant w_2$ .



## Proposition

We have a greedy algorithm that decides comparability efficiently in the m-Cambrian lattice. It consists of reading the letters of  $R(c)^{m+1}$  from left to right, and try to flip each letter in turn.

### Proposition

If the flip root of a covering relation appears in the *i*-th copy of R(c), then only the entry  $w_{m-i}$  of the *m*-noncrossing partition is modified.

### Corollary

The existence of such an increasing *c*-chain between two elements is equivalent to our comparison scheme of  $\leq_{(m)}$ .

# Further directions and open questions

### Corollary

Since the unique *c*-increasing chain of each interval is lexicography smaller than all other chains, the *m*-Cambrian lattices are EL-shellable.

### Corollary

We can efficiently generate the Cambrian lattices thanks to the greedy algorithm, and all m-Cambrian lattices thanks to the new comparison criterion.

### Remark

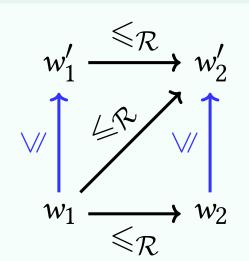
Our proofs are inductive and uniform, involving all Coxeter groups and all choices of Coxeter element at the same time.

- Bousquet-Mélou, M., Fusy, É., and Préville-Ratelle, L.-F. (2012). The number of intervals in the *m*-Tamari lattices. *Electron. J. Comb.*, 18(2):research paper p31, 26.
- Reading, N. (2006). Cambrian lattices.

### Corollary

We can define a binary relation on Cambrian intervals, such that  $[w_1, w_1'] \leqslant [w_2, w_2']$  if  $w_1 \leqslant_{\mathcal{R}} w_2$ ,  $w_1 \leqslant_{\mathcal{R}} w_2'$ , and  $w_2 \leqslant_{\mathcal{R}} w_2'$  in the noncrossing partition lattice  $\mathrm{NCL}(W, c)$ .

This relation is transitive and antisymmetric, and its m-multichains are in bijection with intervals in the m-Cambrian lattice.



Adv. Math., 205(2):313-353.

Stump, C., Thomas, H., and Williams, N. (2020).

Cataland: why the Fuss?

In Proceedings of the 28th international conference on formal power series and algebraic combinatorics, FPSAC 2016, Vancouver, Canada, July 4–8,

### Question

Can we use this "almost" poset structure on Cambrian chains in order to understand the conjecture that there are as many intervals in the linear type-*A m*-Cambrian lattice as in the *m*-Tamari lattice? [Bousquet-Mélou et al., 2012, Stump et al., 2020]

### Question

The noncrossing partition lattice corresponds to the shard order (i.e. core label order) of the Cambrian lattice. Can we generalize this *m*-construction by replacing the Cambrian lattice by some other lattices, e.g. distributive lattices, or semidistributive and congruence-uniform (and trim?) lattices?

2016, pages 1123–1134. Nancy: The Association. Discrete Mathematics & Theoretical Computer Science (DMTCS).

Tamari, D. (1962).

The algebra of bracketings and their enumeration. *Nieuw Arch. Wiskd., III. Ser.*, 10:131–146.