Equivariant γ-positivity of Chow rings and augmented Chow rings of matroids

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Introduction

A polynomial $f(t)$ of degree d is said to be **γ-positive** if it can be expressed as

$$f(t) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_k t^k (1+t)^{d-2k}$$

such that $\gamma_k \geq 0$ for all k. It is not hard to see that γ-positivity implies palindromicity and unimodality of a polynomial.

For a loopless matroid M, the Chow ring $A(M)$ and augmented Chow ring $\widetilde{A}(M)$ satisfy a Poincaré duality and Hard Lefschetz theorem. Consequently, the Hilbert series of $A(M)$ and $\widetilde{A}(M)$ are palindromic and unimodal. Their γ-positivity was later shown by Ferroni, Matherne, Stevens, and Vecchi [FMSV24], and independently by Wang [FMSV24, p.33]. However, no interpretation of the γ_k coefficients was known.

Angarone, Nathanson, Reiner [ANR25] studied the representations of $\mathrm{Aut}(M)$ on Chow rings of matroids for general matroids M and proposed several conjectures. One of them is that the Chow ring $A(M)$ is G-equivariant γ-positive for any $G \leq \mathrm{Aut}(M)$.

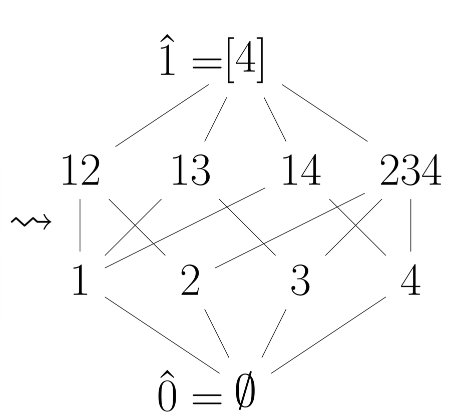
Matroids and matroid Chow rings

A **matroid** M on $[n] := \{1, 2, \ldots, n\}$ of rank r can be though as an abstraction of n vectors that span an r-dimensional vector space.

Example

A matroid M on $[4]$ of rank $r = 3$ and its **lattice of flats** $\mathcal{L}(M)$:

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow$$



The **Chow ring of a (loopless) matroid** M encodes the information of $\mathcal{L}(M)$ and is defined as

$$A(M) := \mathbb{R}[x_F : F \in \mathcal{L}(M) \setminus \{\emptyset\}]/(I + J)$$

where $I = (x_F x_{F'} : F \not\subseteq F',\ F \not\supseteq F')$ and $J = (\sum_{F: i \in F} x_F : i \in [n])$.

Proposition (Feichtner, Yuzvinsky 2003)

The following set of monomials forms a basis for $A(M)$

$$FY(M) := \left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \ldots x_{F_\ell}^{a_\ell} : \begin{smallmatrix} \emptyset = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_\ell \text{ for } 0 \leq \ell \leq n \\ 1 \leq a_i \leq \mathrm{rk}_M(F_i) - \mathrm{rk}_M(F_{i-1}) - 1 \end{smallmatrix} \right\},$$

The action of $G \leq \mathrm{Aut}(M)$ on $\mathcal{L}(M)$ induces a representation of G on $A(M)$; furthermore, $FY(M)$ is a permutation basis under this action.

Conjecture (Angarone, Nathanson, Reiner 2023)

Consider $\mathsf{Hilb}_G(A(M)_{\mathbb{C}}, t) := \sum_{i=0}^{r-1} [A^i_{\mathbb{C}}] t^i$ where $[A^i_{\mathbb{C}}]$ is the isomorphism class of $A^i_{\mathbb{C}}$ in the Grothendieck ring $R_{\mathbb{C}}(G)$ of $\mathbb{C}G$-modules. Then

$$\mathsf{Hilb}_G(A(M)_{\mathbb{C}}, t) = \sum_{i=0}^{\lfloor \frac{r-1}{2} \rfloor} \gamma_i t^i (1+t)^{r-1-2i}$$

and $\gamma_i \in R_{\mathbb{C}}(G)$ is a class of a genuine representation of G for all i.

General equivariant γ-expansion

For $S = \{s_1 < s_2 < \ldots < s_\ell\} \subseteq [r-1]$, consider permutation module generated by chains in $\mathcal{L}(M)$, $\alpha_{\mathcal{L}(M)}(S) := \mathbb{C}G\{F_1 \subsetneq \ldots \subsetneq F_\ell : \mathrm{rk}(F_i) = s_i\ \forall i\}$, and the virtual representation

$$\beta_{\mathcal{L}(M)}(S) := \sum_{T \subseteq S} (-1)^{|S|-|T|} \alpha_{\mathcal{L}(M)}(T).$$

Then $A(M)$ is a direct sum of $\alpha_{\mathcal{L}(M)}(S)$ for some subsets S and

$$\begin{aligned} \mathsf{Hilb}_G(A(M)_{\mathbb{C}}, t) &= \sum_{S \in \mathrm{Stab}([2,r-1])} \phi_{S,r}(t) [\alpha_{\mathcal{L}(M)}(S)] \\ &= \sum_{T \in \mathrm{Stab}([2,r-1])} \left(\sum_{T \subseteq S \subseteq [r-1]} \phi_{S,r}(t) \right) [\beta_{\mathcal{L}(M)}(T)] \end{aligned}$$

where $\mathrm{Stab}(S)$ *is the collection of subsets of S containing no consecutive integers* and

$$\phi_{S,r}(t) = t^{|S|} [s_1 - 1]_t [s_2 - s_1 - 1]_t \ldots [s_\ell - s_{\ell-1} - 1]_t [r - s_\ell]_t.$$

Lemma (L. 2024)

For $n \geq 2$ and any subset $T \in \mathrm{Stab}([2, n-1])$,

$$\sum_{T \subseteq S \subseteq [n-1]} \phi_{S,n}(t) = t^{|T|} (1+t)^{n-1-2|T|},$$

By a theorem of Stanley, $\beta_{\mathcal{L}(M)}(S) \cong_G \tilde{H}_{|S|-1}(\mathcal{L}(M)_S)$.

Theorem 1. (L. 2024)

For $G \leq \mathrm{Aut}(M)$, both $A(M)$ and $\widetilde{A}(M)$ are G-equivariant γ-positive:

$$\mathsf{Hilb}_G(A(M)_{\mathbb{C}}, t) = \sum_{S \in \mathrm{Stab}([2,r-1])} [\tilde{H}_{|S|-1}(\mathcal{L}(M)_S)] t^{|S|} (1+t)^{r-1-2|S|}$$

$$\mathsf{Hilb}_G(\widetilde{A}(M)_{\mathbb{C}}, t) = \sum_{S \in \mathrm{Stab}([r-1])} [\tilde{H}_{|S|-1}(\mathcal{L}(M)_S)] t^{|S|} (1+t)^{r-2|S|}$$

γ-expansion for uniform matroids $U_{r,n}$

For a **uniform matroid** $U_{r,n}$ of rank r, the Chow rings carry representations of $\mathrm{Aut}(U_{r,n}) = \mathfrak{S}_n$. We encode $\mathfrak{S}_n$-representations in terms of symmetric functions via the **Frobenius characteristic map** ch. For a graded $\mathbb{C}\mathfrak{S}_n$-module $V = \bigoplus_i V_i$, the **graded Frobenius series** of V is

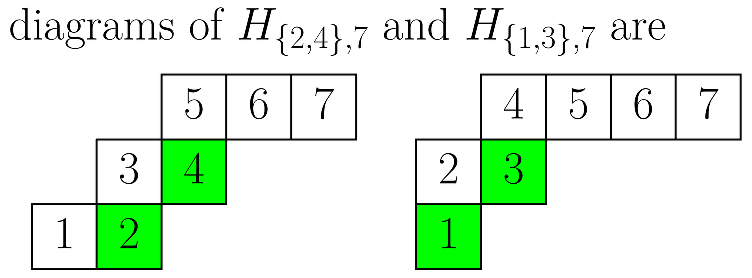
$$\mathsf{grFrob}(V, t) := \sum_i \mathrm{ch}(V_i) t^i.$$

Ribbon Schur function

A **ribbon Schur function** $s_{H_{R,n}}$ is a Schur function indexed by $H_{R,n}$ for $R \subseteq [n-1]$.

Example

Let $n = 7$. Then the diagrams of $H_{\{2,4\},7}$ and $H_{\{1,3\},7}$ are



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Corollary 2 (L. 2024)

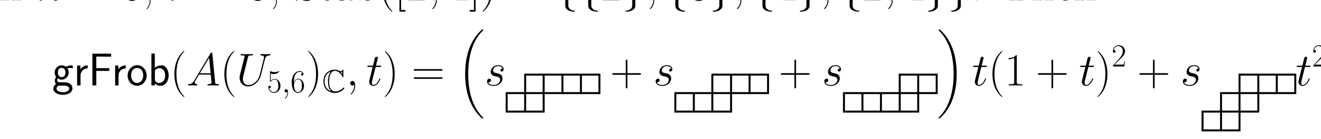
For any positive integer n and $1 \leq r \leq n$,

1. $\mathsf{grFrob}(A(U_{r,n})_{\mathbb{C}}, t) = \sum_{R \in \mathrm{Stab}([2,r-1])} s_{H_{R,n}} t^{|R|} (1+t)^{r-1-2|R|}$.
2. $\mathsf{grFrob}(\widetilde{A}(U_{r,n})_{\mathbb{C}}, t) = \sum_{R \in \mathrm{Stab}([r-1])} s_{H_{R,n}} t^{|R|} (1+t)^{r-2|R|}$

In particular, $\dim \mathrm{ch}^{-1}(s_{H_{R,n}}) = |\{\sigma \in \mathfrak{S}_n :\ \mathsf{DES}(\sigma) = R\}$.

Example

When $n = 6$, $r = 5$, $\mathrm{Stab}([2,4]) = \{\{2\}, \{3\}, \{4\}, \{2,4\}\}$. Then



When $r = n$, Corollary 2 recovers Shareshian and Wachs' Schur-γ-positivity of the Eulerian quasisymmetric functions.

The irreducible decompositions of $A(U_{r,n})$ and $\widetilde{A}(U_{r,n})$

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For $P \in SYT(\lambda)$, $\mathsf{DES}(P) := \{i \in [n-1] :\ i+1 \text{ appears in a lower row than } i\}$.

Let $\mathsf{grFrob}(A(U_{r,n})_{\mathbb{C}}, t) = \sum_{\lambda \vdash n} P^r_\lambda(t) s_\lambda$ and $\mathsf{grFrob}(\widetilde{A}(U_{r,n})_{\mathbb{C}}, t) = \sum_{\lambda \vdash n} \widetilde{P}^r_\lambda(t) s_\lambda$.

Corollary 3 (L. 2024)

For $\lambda \vdash n$, we have

1. $$P^r_\lambda(t) = \sum_{\substack{P \in SYT(\lambda) \\ \mathsf{DES}(P) \in \mathrm{Stab}([2,r-1])}} t^{\mathsf{des}(P)} (1+t)^{r-1-2\mathsf{des}(P)} = \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \xi_{r,\lambda,k}\, t^k (1+t)^{r-1-2k}$$
2. $$\widetilde{P}^r_\lambda(t) = \sum_{\substack{P \in SYT(\lambda) \\ \mathsf{DES}(P) \in \mathrm{Stab}([r-1])}} t^{\mathsf{des}(P)} (1+t)^{r-2\mathsf{des}(P)} = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor} \widetilde{\xi}_{r,\lambda,k}\, t^k (1+t)^{r-2k}$$

where

- $\xi_{r,\lambda,k} = |\{P \in SYT(\lambda) : \mathsf{DES}(P) \in \mathrm{Stab}([2, r-1]), \mathsf{des}(P) = k\}$
- $\widetilde{\xi}_{r,\lambda,k} = |\{P \in SYT(\lambda) : \mathsf{DES}(P) \in \mathrm{Stab}([r-1]), \mathsf{des}(P) = k\}$

In particular, $P^r_\lambda(t)$ and $\widetilde{P}^r_\lambda(t)$ are palindromic and unimodal for all λ and r.

Acknowledgements

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