The (q,t)-tau functions and path operators

Houcine Ben Dali [1] Valentin Bonzom [2] Maciej Dołęga [3]

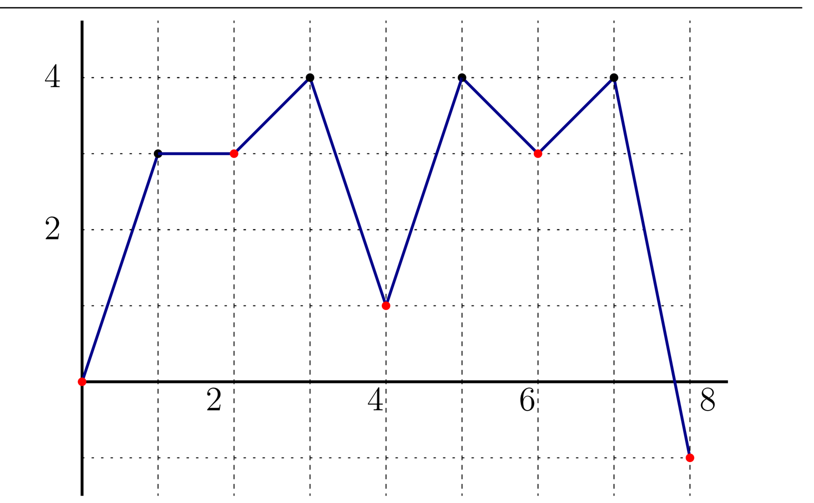
[1]Harvard University [2]Université Gustave Eiffel [3]Polish Academy of Sciences

Main results

- We introduce a family of combinatorial differential operators on the space of symmetric function, which are described by lattice paths.
- We show that path operators are representation of some elements in the Shuffle algebra called Negut elements.
- We use these operators to provide a family of PDEs which characterize the (q,t)-tau function.

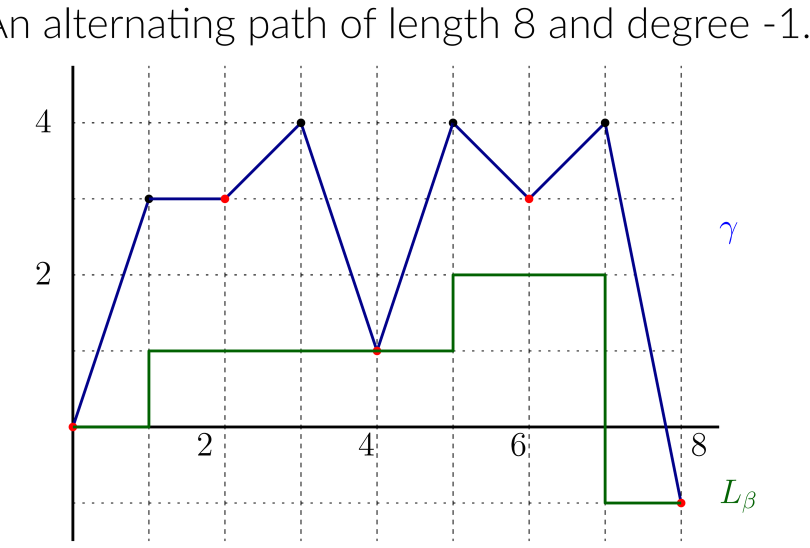
Alternating Paths

An alternating path of **length** $2\ell > 0$ and **degree** $n \in \mathbb{Z}$, is a path in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$, starting at $(0,0)$ ending at $(2\ell, n)$, and such that an odd (resp. even) step is a weakly up step (resp. weakly down step).
A valley is a point of the path with an even x-coordinate. The other points are peaks.



An alternating path of length 8 and degree -1.

Fix a sequence $\beta := (\beta_1, \dots, \beta_\ell) \in \mathbb{Z}^\ell$. Define L_β as the path starting at $(0,0)$ ending at $(2\ell, |\beta|)$ and with vertical increment β_j at x-coordinate $2j-1$.
Define $\mathbf{R}_\beta$ as the set of alternating paths of length 2ℓ, degree $|\beta|$, staying weakly above L_β.
If $\gamma \in \mathbf{R}_\beta$ and V is a valley of γ, we define its β-height $\mathrm{ht}_\beta(V) \geq 0$ as the height w.r.t to L_β.
The five valleys of the path in the example have respective β-heights $0, 2, 0, 1, 0$.



A path $\gamma \in \mathbf{R}_\beta$ for $\beta = (1, 0, 1, -3)$.

Path operators

We associate to each one-step path of degree k an operator:

$$\mathcal{O}(k) := \begin{cases} h_k[-X] & \text{if } k > 0 \\ h_k^{\perp}[MX] & \text{if } k < 0 \\ 1 & \text{if } k = 0, \end{cases}$$

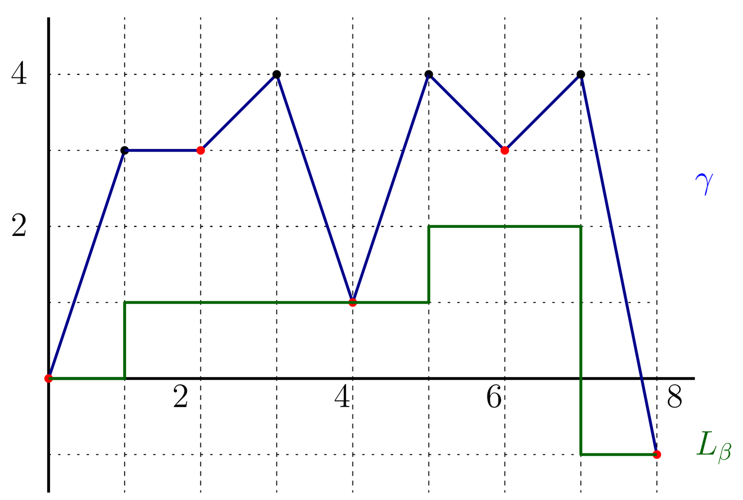
To a path $\gamma \in \mathbf{R}_\beta$ with steps $(\gamma_1, \dots, \gamma_{2\ell})$, we associate the operator

$$\mathcal{O}_\beta(\gamma) := \left(\prod_{V \text{valley}} (qt)^{\mathrm{ht}_\beta(V)} \right) \mathcal{O}(\gamma_1) \dots \mathcal{O}(\gamma_{2\ell}).$$

If γ has degree n, then $\mathcal{O}_\beta(\gamma)$ is homogeneous of degree n.

Example: The operator associated to the path γ above is

$$\mathcal{O}_\beta(\gamma) = (qt)^2 (qt)^1 h_3[-X] h_1[-X] h_3^{\perp}[MX] h_3[-X] h_1^{\perp}[MX] h_1[-X] h_5^{\perp}[MX].$$



A path $\gamma \in \mathbf{R}_\beta(1, 0, 1, -3)$.

Define the operator $\mathcal{R}_\beta := \sum_{\gamma \in \mathbf{R}_\beta} \mathcal{O}_\beta(\gamma)$.

Connection to the shuffle algebra and Negut elements

Theorem (B.D.–Bonzom–Dołęga) Fix $\beta = (\beta_1, \dots, \beta_\ell) \in \mathbb{Z}^\ell$, then

$$\mathcal{R}_\beta = [z_1^{\beta_1} \cdots z_\ell^{\beta_\ell}] \frac{D(z_1) \cdots D(z_\ell)}{\prod_{i=1}^{\ell-1} (1 - qt z_{i+1}/z_i)},$$

where

$$D(z) = \sum_{m,n \geq 0} z^{m-n} h_m[-X] h_n^{\perp}[MX].$$

This formula can be used to show that the path operators $\mathcal{R}_\beta$ coincide with some element of the **shuffle algebra** introduced by Negut [4].

Reparametrization

We consider now sequences $\beta = (\beta_1, \dots, \beta_\ell) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}^{\ell-1}$.

We decorate peaks of alternating paths by **particles**: one particle corresponds to a unit increment of L_β. In other terms, there are β_i particles on the i-th peak.

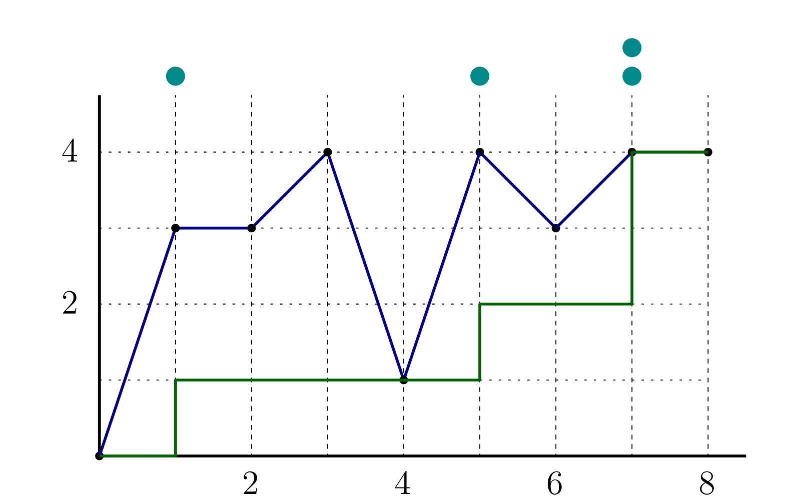
Fix $\beta \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geq 0}^{\ell-1}$ of size n, we define a sequence $\alpha \in \mathbb{Z}_{\geq 0}^n$ such that:

- $2\alpha_i$ is the distance between the i-th and the $i+1$-th particles, for $1 \leq i \leq n-1$,
- $2\alpha_n$ is the distance between the last particle and the last peak.

We then define the set $\mathbf{Q}_\alpha$ by

$$\mathbf{Q}_\alpha := \mathbf{R}_\beta$$

and the operator $\mathcal{Q}_\alpha := \mathcal{R}_\beta = \sum_{\gamma \in \mathbf{Q}_\alpha} \mathcal{O}_\beta(\gamma)$.



An alternating path γ in $\mathbf{Q}_{(2,1,0,0)} = \mathbf{R}_{(1,0,1,2)}$.

The operators $\mathcal{A}_G^{(n)}$

Definition. If $G(\hbar) = u_0 + u_1\hbar + u_2\hbar^2 + \dots$, we define for $n \geq 1$ the operator

$$\mathcal{A}_G^{(n)} := \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} u_{\alpha_1} \dots u_{\alpha_n} \mathcal{Q}_\alpha,$$

defined as formal power-series in the variables u_i.
→ the series of all path operators of degree n with an extra weight u_i for two particles separated by distance i.

The (q,t)-tau function

Fix two formal power-series G_1 and G_2 (in the variable $\hbar$):

$$G_1(\hbar) := 1 + \sum_{n=1}^{\infty} u_n \hbar^n, \quad G_2(\hbar) := 1 + \sum_{n=1}^{\infty} v_n \hbar^n.$$

And let $G(\hbar) := \frac{G_1(\hbar)}{G_2(\hbar)}$.

We define the G-**weighted** (q,t)-**tau function** by:

$$\tau_G(z, X, Y) := \sum_{\lambda \text{ partition}} z^{|\lambda|} \frac{\widetilde{H}_\lambda^{(q,t)}[X] \widetilde{H}_\lambda^{(q,t)}[Y]}{\left\| \widetilde{H}_\lambda^{(q,t)} \right\|_*^2} \prod_{(i,j) \in \lambda} G(q^{j-1} t^{i-1}),$$

where $\widetilde{H}_\lambda^{(q,t)}$ denotes the modified Macdonald polynomial.

- This function is a natural (q,t)-deformation of the tau function for the classical G-weighted Hurwitz numbers, as well as the b-Hurwitz numbers introduced by Chapuy and Dołęga.
- It is conjectured [3] to be related (for some weights G) to the generating series of the mixed Hodge polynomials of character varieties of the Riemann sphere.

Theorem (B.D.–Bonzom–Dołęga) For any $n \geq 1$ we have

$$z^n \mathcal{A}_{G_1}^{(n)}(X) \cdot \tau_G(z, X, Y) = \left(\mathcal{A}_{G_2}^{(n)}(Y) \right)^* \cdot \tau_G(z, X, Y),$$

where $\left(\mathcal{A}_{G_2}^{(n)} \right)^*$ is the adjoint of $\mathcal{A}_{G_2}^{(n)}$. These equations fully characterize the function $\tau_G(z, X, Y)$. Moreover,

$$\tau(z, X, Y) = \sum_{\lambda \text{ partition}} z^{|\lambda|} \mathfrak{a}_{G_1,\lambda}(X) \mathfrak{b}_{G_2,\lambda}(Y),$$

where $\mathfrak{a}_{G,\lambda} := \mathcal{A}_G^{(\lambda_1)} \dots \mathcal{A}_G^{(\lambda_{\ell(\lambda)})} \cdot 1$ is a basis of the space of symmetric functions, and $(\mathfrak{b}_{G,\lambda})$ is its dual basis.

Proof: We show that the operators $\mathcal{Q}_\alpha$ satisfy a family of commutation relations:

$$\mathcal{Q}_{(n+1)} = \frac{1}{M} \left[D_0, \mathcal{Q}_{(n)} \right], \quad \text{for any } n \geq 0,$$

and

$$\sum_{\sigma \in \mathfrak{S}_n} \mathcal{Q}_{\sigma(\alpha)} = \frac{1}{M} \sum_{\sigma \in \mathfrak{S}_n} \left[\mathcal{Q}_{\alpha_{\sigma(n)}-1}, \mathcal{Q}_{\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n-1)}} \right] \quad \text{for any } \alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n.$$

References

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FPSAC 2025, Hokkaido University, Japan

bendali@math.harvard.edu, valentin.bonzom@univ-eiffel.fr, mdolega@impan.pl