2025.7 FPSAC Hokkaido University, Sapporo

Boolean structure constants

Yibo Gao[1] and Hai Zhu[2]

[1]Beijing International Center for Mathematical Research, Peking University; [2]School of Mathematical Sciences, Peking University

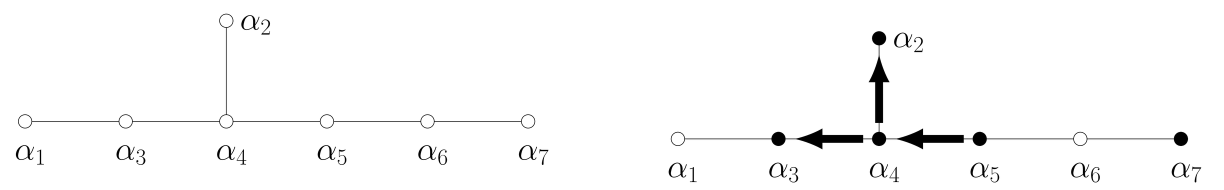
Motivation

- The Schubert problem asks for combinatorial interpretations of the structure constants $c_{u,v}^w \in \mathbb{Z}_{\geq 0}$ in the expansion $\sigma_u \cdot \sigma_v = \sum_w c_{u,v}^w \sigma_w$ with respect to the Schubert classes $\sigma_w \in H^*(G/B;\mathbb{Z})$.
- Boolean elements plays a crucial role in the study of Schubert calculus. For instance, the Schubert variety X_w is a toric variety if and only if w is boolean (4), and the Schubert variety $X_{w_0(I)c}$ is L_I-spherical if and only if c is boolean (2, 3).

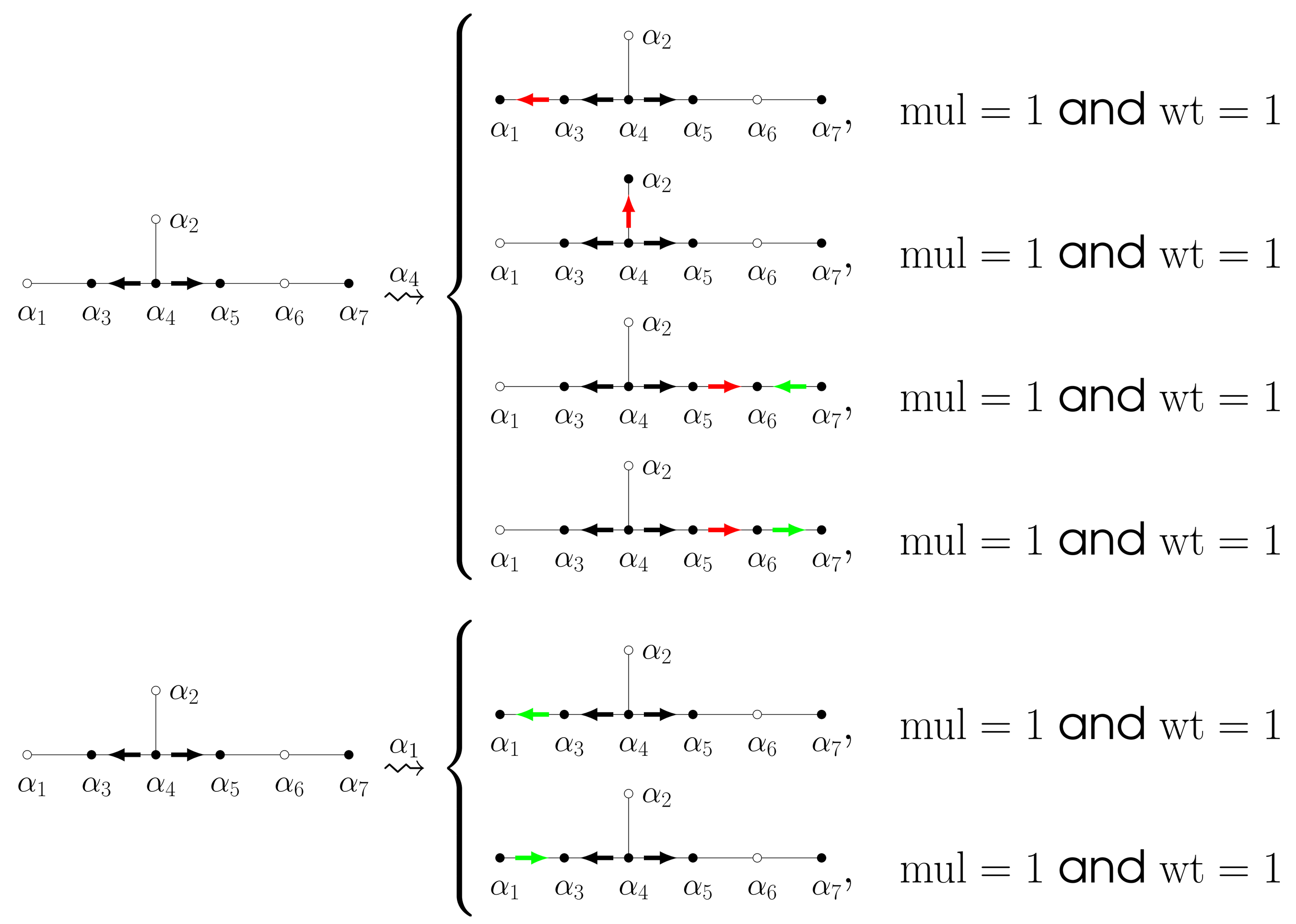
Goal: Make progress towards the Schubert problem in the boolean case.

Background

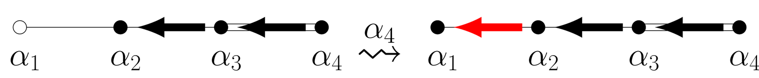
- A Weyl group element $w \in W$ is **boolean** if and only if w is a product of distinct simple reflections.
- The **boolean diagram** $B(w)$ of a boolean element $w \in W$: e.g. $w = s_3s_2s_4s_5s_7 \in W(E_7)$.



- Define two kinds of **boolean insertions** and their **multiplicities** $\mathrm{mul}()$ and **weights** $\mathrm{wt}()$ as follows:
  - **Non-equivariant boolean insertions:**

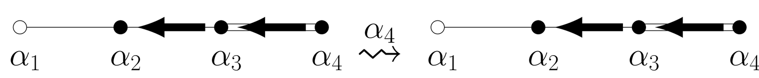


For the Dynkin diagram of type C_4, there are **two** edges from α_4 to α_3 and one edge from α_3 to α_4.



α_4, $\mathrm{mul} = \mathbf{2}$ and $\mathrm{wt} = 1$

  - **Equivariant boolean insertions:**



α_4, $\mathrm{mul} = 1$, $\mathrm{wt} = \mathbf{2t_2 + 2t_3 + t_4}$

where $t_\alpha := \omega_\alpha - s_\alpha(\omega_\alpha)$.

Main Results

Let G be a complex, connected, reductive algebraic group and B be a Borel subgroup of G with a maximal torus T and the Weyl group $W = N_G(T)/T$.

Theorem 1 (Torus equivariant cohomology version) *For boolean elements $u, v, w \in W$,*

$$d_{u,v}^w = \begin{cases} \sum_{u \overset{S(v)}{\rightsquigarrow} w} \mathrm{mul}(u \overset{S(v)}{\rightsquigarrow} w) \cdot \mathrm{wt}(u \overset{S(v)}{\rightsquigarrow} w), & \text{if there exists a boolean insertion path } v \overset{S(u)}{\rightsquigarrow} w \\ 0, & \text{otherwise} \end{cases}$$

where the summation is over all boolean insertion paths $u \overset{S(v)}{\rightsquigarrow} w$.

Corollary 1 (Cohomology version) *For boolean elements $u, v, w \in W$,*

$$c_{u,v}^w = \begin{cases} \sum_{u \overset{S(v)}{\rightsquigarrow} w} \mathrm{mul}(u \overset{S(v)}{\rightsquigarrow} w), & \text{if there exists a non-equivariant boolean insertion path } v \overset{S(u)}{\rightsquigarrow} w \\ 0, & \text{otherwise} \end{cases}$$

where the summation is over all non-equivariant boolean insertion paths $u \overset{S(v)}{\rightsquigarrow} w$.

The implementation of Corollary 1 yields a **multiplicity-free** result in type A.

Corollary 2 (Multiplicity-free) *For boolean elements u, v, w in the Weyl group of type A, $c_{u,v}^w = 1$ if there exist non-equivariant boolean insertion paths $u \overset{S(v)}{\rightsquigarrow} w$ and $v \overset{S(u)}{\rightsquigarrow} w$; $c_{u,v}^w = 0$ otherwise.*

Proof of Theorem 1 (Sketch)

- **Step 1:** The equivariant Chevalley's formula (1, p.351, Theorem 19.1.2) indicates that

$$[\xi_w]\Big(\xi_u \prod_{\beta \in S} \xi_{s_\beta}\Big) = \sum_{u \overset{S}{\rightsquigarrow} w} \mathrm{mul}(u \overset{S}{\rightsquigarrow} w)\,\mathrm{wt}(u \overset{S}{\rightsquigarrow} w).$$

- **Step 2:** A very unique property of boolean elements: If $u \overset{S(v)}{\rightsquigarrow} w$ and $v \overset{S(u)}{\rightsquigarrow} w$, then

$$[\xi_w](\xi_u \cdot \xi_v) = [\xi_w]\Big(\xi_u \prod_{\beta \in S(v)} \xi_{s_\beta}\Big) = [\xi_w]\Big(\Big(\prod_{\alpha \in S(u)} \xi_{s_\alpha}\Big)\xi_v\Big) = [\xi_w]\Big(\Big(\prod_{\alpha \in S(u)} \xi_{s_\alpha}\Big)\Big(\prod_{\beta \in S(v)} \xi_{s_\beta}\Big)\Big).$$

Reason: For distinct boolean pairs (u', v') and (u'', v'') such that $S(u') = S(u'')$ and $S(v') = S(v'')$, the Schubert expansions of $\xi_{u'} \cdot \xi_{v'}$ and $\xi_{u''} \cdot \xi_{v''}$ share **no** common boolean terms.

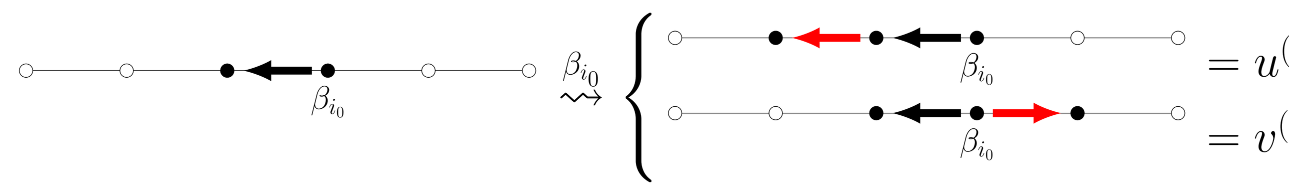
Proof of Corollary 2 (Sketch)

It suffices to show the following fact: In type A_m, fix an ordering $S = \{\beta_1, \cdots, \beta_n\}$ of a set of simple roots $S \subseteq \Delta$, then there exists **at most one** non-equivariant boolean insertion path $u \overset{S}{\rightsquigarrow} w$ for any boolean elements $u, w \in W$.

Assume that there are two distinct non-equivariant boolean insertion paths $u = u^{(0)} \overset{\beta_1}{\rightsquigarrow} u^{(1)} \overset{\beta_2}{\rightsquigarrow} \cdots \overset{\beta_n}{\rightsquigarrow} u^{(n)} = w$ and $u = v^{(0)} \overset{\beta_1}{\rightsquigarrow} v^{(1)} \overset{\beta_2}{\rightsquigarrow} \cdots \overset{\beta_n}{\rightsquigarrow} v^{(n)} = w$ which firstly differ at the i_0-th step.

- **Case 1:** $B(u^{(i_0)})$ and $B(v^{(i_0)})$ possess reverse directions on some common edges.
- **Case 2:**

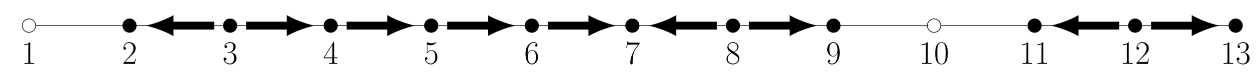
$u^{(i_0-1)} = v^{(i_0-1)} =$



$u^{(i_0)}$ has more vertices than $v^{(i_0)}$ on the left of β_{i_0}. So do $u^{(i)}$, $v^{(i)}$ and β_{i_0} whenever $i \geq i_0$.

Fast Algorithms

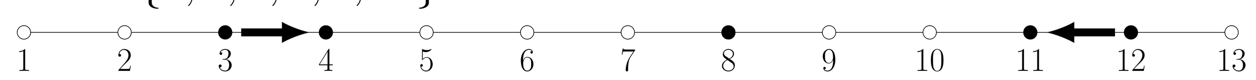
- **Goal:** In type A_{13}, let $u = s_4s_3s_8s_{11}s_{12}$, $S = \{2,3,6,7,8,12\}$ and $w = s_7s_6s_5s_4s_2s_3s_9s_8s_{11}s_{13}s_{12}$. Figure out a boolean insertion path $u \overset{S}{\rightsquigarrow} w$ if exists.
- **Runtime:** $O(n^2)$.
- $B(w)$:



Each insertion should yield a new boolean diagram **contained** in $B(w)$.

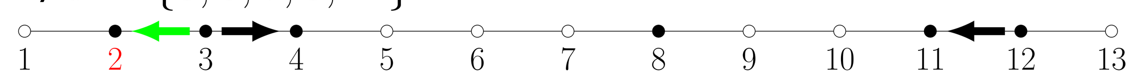
- **Initialize:** $S = \{2,3,6,7,8,12\}$

$B(u^{(0)})$:



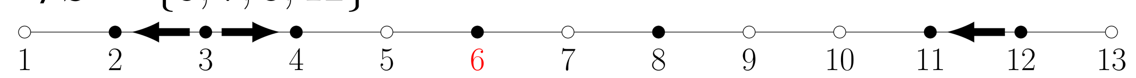
- **Step 1:** Insert all the vertices i such that all the possible boolean insertions $B \overset{i}{\rightsquigarrow}$ can only add a unique vertex into B.
  - $u^{(0)} \overset{2}{\rightsquigarrow} u^{(1)}$, $S = \{3,6,7,8,12\}$

$B(u^{(1)})$:



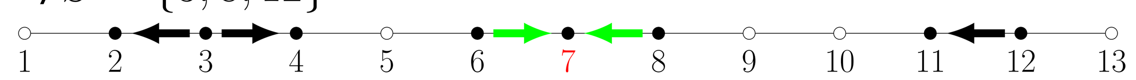
  - $u^{(1)} \overset{6}{\rightsquigarrow} u^{(2)}$, $S = \{3,7,8,12\}$

$B(u^{(2)})$:



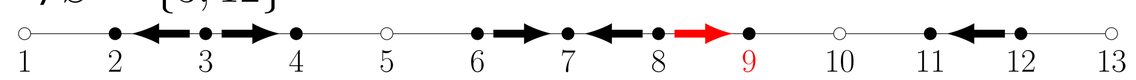
  - $u^{(2)} \overset{7}{\rightsquigarrow} u^{(3)}$, $S = \{3,8,12\}$

$B(u^{(3)})$:



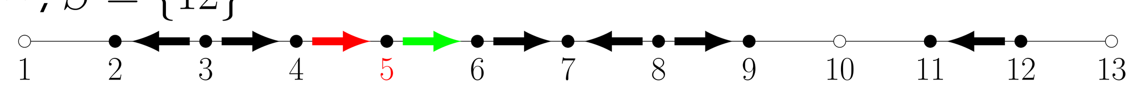
  - $u^{(3)} \overset{8}{\rightsquigarrow} u^{(4)}$, $S = \{3,12\}$

$B(u^{(4)})$:



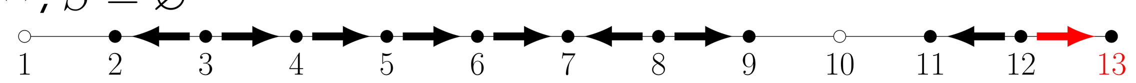
- **Step 2:** Write $S = \{i_1 < \cdots < i_m\}$ ($S = \{3,12\}$ in our example) and $B(w) \setminus B = \{j_1 < \cdots < j_m\}$ ($B(w) \setminus B = \{5,13\}$ in our example). Do $B \overset{i_k}{\rightsquigarrow} B'$ ($k = 1, \cdots, m$) such that the newly added vertex in B' is exactly j_k and that $B' \subseteq B(w)$ if possible.
  - $u^{(4)} \overset{3}{\rightsquigarrow} u^{(5)}$, $S = \{12\}$

$B(u^{(5)})$:



  - $u^{(5)} \overset{12}{\rightsquigarrow} u^{(6)}$, $S = \varnothing$

$B(u^{(6)})$:



- **Result:** A boolean insertion path

$$u = u^{(0)} \overset{2}{\rightsquigarrow} u^{(1)} \overset{6}{\rightsquigarrow} u^{(2)} \overset{7}{\rightsquigarrow} u^{(3)} \overset{8}{\rightsquigarrow} u^{(4)} \overset{3}{\rightsquigarrow} u^{(5)} \overset{12}{\rightsquigarrow} u^{(6)} = w.$$

References

(1) David Anderson and William Fulton. *Equivariant cohomology in algebraic geometry*. Cambridge University Press, 2023.

(2) Yibo Gao, Reuven Hodges, and Alexander Yong. "Classification of Levi-spherical Schubert varieties". In: *Selecta Math. (N.S.)* 29.4 (2023), Paper No. 55, 40. DOI: 10.1007/s00029-023-00856-9.

(3) Yibo Gao, Reuven Hodges, and Alexander Yong. "Levi-spherical Schubert varieties". In: *Adv. Math.* 439 (2024), Paper No. 109486, 14. DOI: 10.1016/j.aim.2024.109486.

(4) Paramasamy Karuppuchamy. "On Schubert varieties". In: *Comm. Algebra* 41.4 (2013), pp. 1365–1368. DOI: 10.1080/00927872.2011.635620.