Random Combinatorial Billiards & Stoned Exclusion Processes

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Abstract

We introduce and study several *random combinatorial billiard trajectories*. Such a system, which depends on a fixed parameter $p \in (0,1)$, models a beam of light that travels in a Euclidean space, occasionally randomly reflecting off of a hyperplane in the Coxeter arrangement of an affine Weyl group with some probability that depends on the side of the hyperplane that it hits. In one case, we (essentially) recover Lam's reduced random walk in the limit as p tends to 0. The investigation of our random billiard trajectories relies on an analysis of new finite Markov chains that we call *stoned exclusion processes*. These processes have remarkable stationary distributions determined by well-studied polynomials such as ASEP polynomials, inhomogeneous TASEP polynomials, and open boundary ASEP polynomials; in many cases, it was previously not known how to construct Markov chains with these stationary distributions. Using multiline queues, we analyze correlations in the *stoned multispecies TASEP*, allowing us to determine limit directions for reduced random billiard trajectories and limit shapes for new random growth processes for n-core partitions.

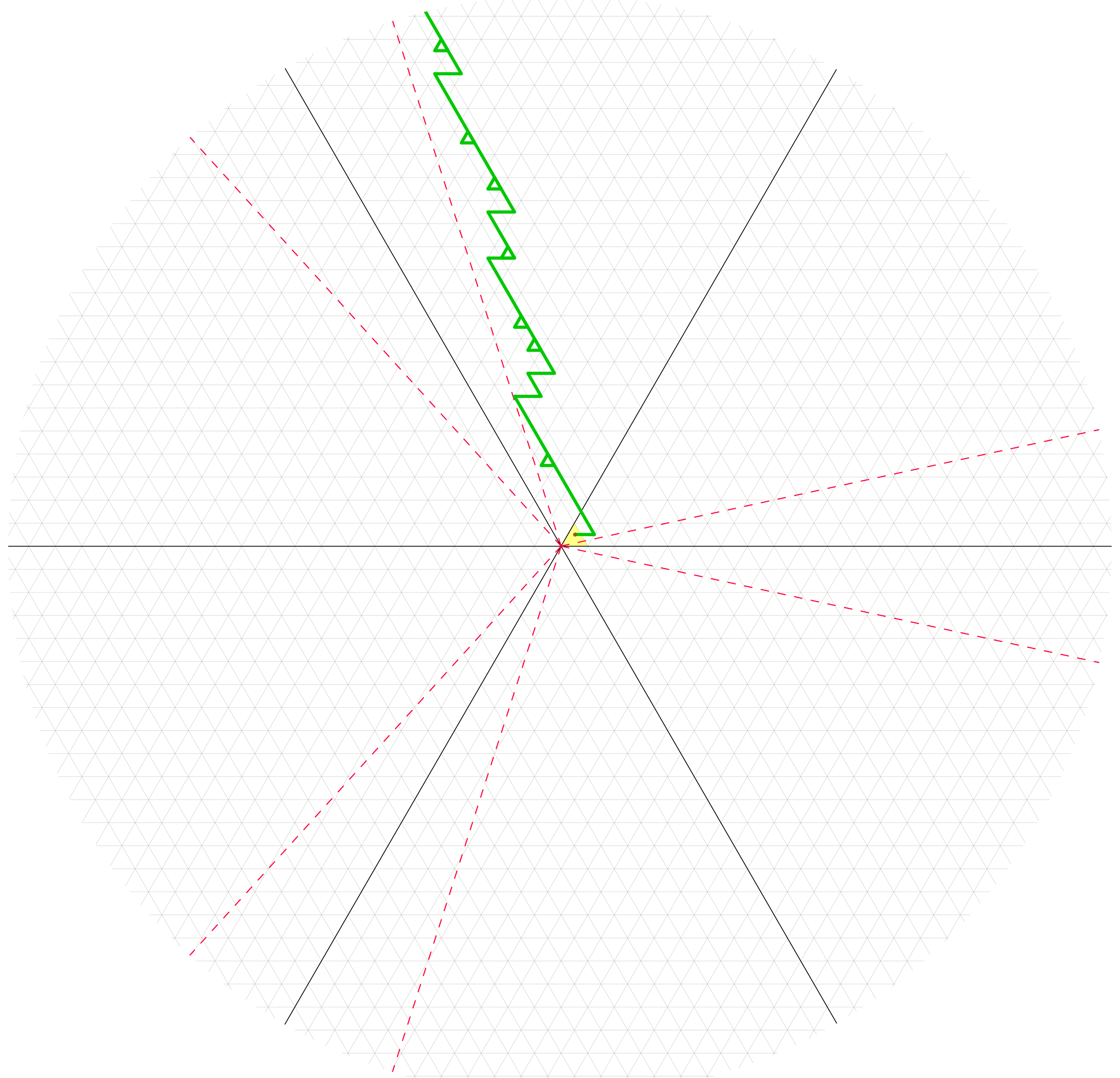
Random Combinatorial Billiards

Let $\widetilde{W}$ be an irreducible affine Weyl group with associated finite Weyl group W. Let $\mathcal{H}_{\widetilde{W}}$ be the Coxeter arrangement of $\widetilde{W}$.

Start at a generic point in the fundamental alcove. Shine a beam of light in some initial rational direct η. If the beam travels in a straight line, then it will pass through a sequence of alcove facets labeled by simple reflections $s_{i_1}, s_{i_2}, s_{i_3}, \ldots$. This sequence is periodic with some period N_η.

Define the *reduced random billiard trajectory* as follows. Fix $p \in (0,1)$. Whenever the beam hits a hyperplane in $\mathcal{H}_{\widetilde{W}}$ that it **has not** yet crossed, it passes through with probability p and reflects with probability $1-p$. Whenever the beam hits a hyperplane in $\mathcal{H}_{\widetilde{W}}$ that it **has** crossed, it reflects. Alternatively, the location of the beam of light after it has hit a hyperplane for the M-th time is the alcove corresponding to the Demazure product $s_{i_M} \star \cdots \star s_{i_2} \star s_{i_1}$.

Theorem. *There exists a vector ψ_η^p such that with probability 1, the reduced random billiard trajectory travels asymptotically in the direction of a vector in $W\psi_\eta^p$. The vector ψ_η^p can be computed explicitly in terms of the stationary distribution of a Markov chain $\mathbf{M}_\eta$ with state space $W \times \mathbb{Z}/N_\eta\mathbb{Z}$.*



Type A

Assume $W = A_{n-1} = \mathfrak{S}_n$. Let $V = \{(\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n : \gamma_1 + \cdots + \gamma_n = 0\}$. Then $\mathcal{H}_{\widetilde{W}} = \{H_{i,j}^k : 1 \le i < j \le n,\ k \in \mathbb{Z}\}$, where $H_{i,j}^k = \{(\gamma_1, \ldots, \gamma_n) \in V : \gamma_i - \gamma_j = k\}$. Let $\delta^{(n)} = (1, 1, \ldots, 1, -(n-1)) \in V$. Let e_i be the i-th standard basis vector in $\mathbb{R}^n$.

Theorem. *The vector $\psi_{\delta^{(n)}}^p$ is a scalar multiple of*

$$\sum_{1 \le i < j \le n} \frac{(j-i)(2n-(i+j-1)p)}{(n-ip)(n-(i-1)p)(n-jp)(n-(j-1)p)}(e_i - e_j).$$

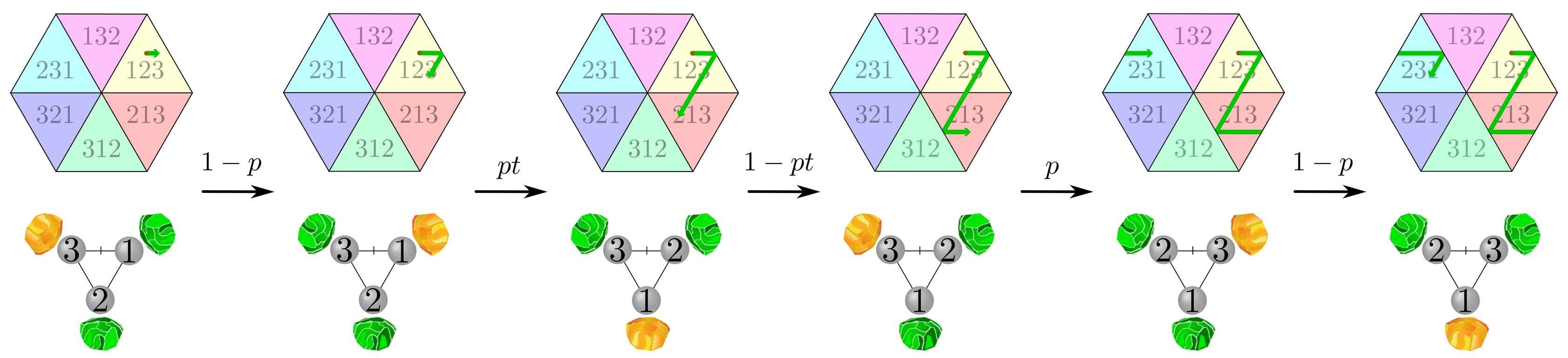
In the limit $p \to 0$, the reduced random billiard trajectory recovers Lam's reduced random walk [8], and the preceding theorem recovers a result of Ayyer and Linusson [1] (originally conjectured by Lam).

The Stoned Multispecies ASEP

Fix $t \in [0,1)$ and $p \in (0,1)$. Fix $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$. Let

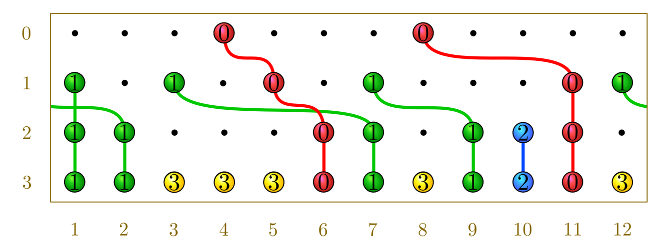
$$\mathfrak{f}_t(k,k') = \begin{cases} 1 & \text{if } k > k'; \\ t & \text{if } k < k'; \\ 0 & \text{if } k = k'. \end{cases}$$

Consider a Markov chain with state space $S_n \times \mathbb{Z}/n\mathbb{Z}$. Represent (μ, j) by placing particles of species $\mu_1, \ldots, \mu_n$ on the sites of a cycle, placing a gold stone on site j, and placing green stones on all other sites.



For a transition from state (μ, j), the gold stone swaps with the green stone on site $j+1$. The stones send a signal to the particles on sites j and $j+1$, telling them to swap. The signal reaches the particles with probability p. If the particles receive the signal, they follow their orders with probability $\mathfrak{f}_t(\mu_j, \mu_{j+1})$.

Corteel, Mandelshtam, and Williams [6], building off of work of Cantini, de Gier, and Wheeler [5], introduced *ASEP polynomials*, which are polynomials $F_\mu(x_1, \ldots, x_n; q, t) \in \mathbb{C}(q,t)[x_1, \ldots, x_n]$. They showed that the stationary probability of a state μ in the multispecies ASEP is $F_\mu(1, \ldots, 1; 1, t)$ (up to normalization). They also gave a combinatorial formula for ASEP polynomials using multiline queues. ASEP polynomials are closely related to MacDonald polynomials.



Theorem. *Let $\chi = \frac{1-p}{1-pt}$. The stationary probability of (μ, j) in the stoned multispecies ASEP is $F_\mu(1, \ldots, 1, \chi, 1, \ldots, 1; 1, t)$ (up to normalization), where the χ is in position j.*

There is a more general version of the stoned multispecies ASEP in which the green stones are numbered $2, \ldots, n$ and the probability of the signal reaching the particles is some probability p_i depending on the number i of the green stone that swapped with the gold stone. In this setting, the stationary distribution of (μ, σ) is

$$F_\mu(\chi_{\sigma^{-1}(1)}, \ldots, \chi_{\sigma^{-1}(n)}; 1, t),$$

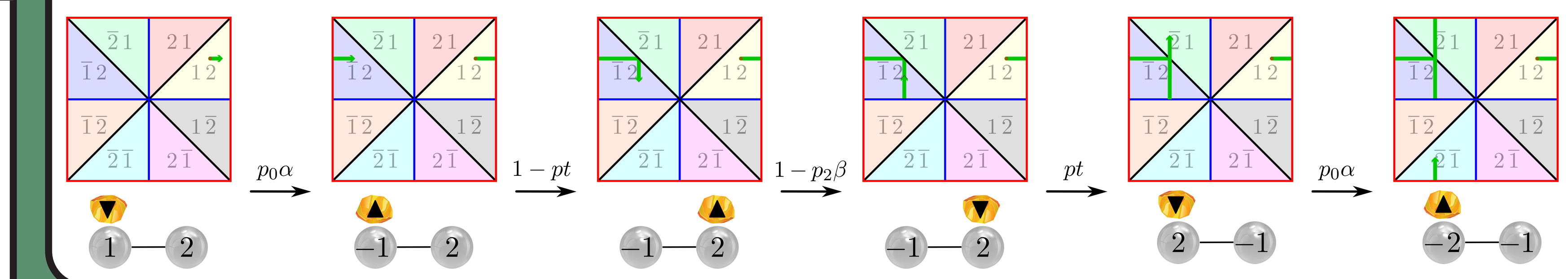
where $\chi_1 = 1$ and $p_j = \frac{\chi_1 - \chi_j}{t\chi_1 - \chi_j}$ for $2 \le j \le n$.

Ayyer, Martin, and Williams [2] recently introduced a *different* Markov chain whose stationary distribution is determined by ASEP polynomials evaluated at generic values.

Stoned Exclusion Processes

Lam and Williams [9] introduced the *inhomogeneous TASEP* and conjectured that its stationary distribution is closely related to Schubert polynomials. Cantini [3] showed that the stationary distribution is governed by particular specializations of *inhomogeneous TASEP polynomials*. In [7], I introduce the *stoned inhomogeneous TASEP* and show that its stationary distribution is given by inhomogeneous TASEP polynomials evaluated at generic values. In a special case, the stoned inhomogeneous TASEP can be interpreted as a random combinatorial billiard trajectory in a torus.

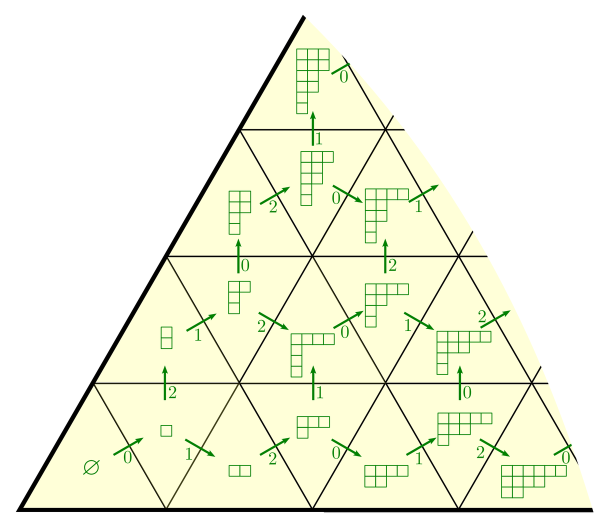
The *multispecies open boundary ASEP* is a famous interacting particle system on a path graph. Cantini, Garbali, de Gier, and Wheeler [4] found that its stationary distribution is governed by particular specializations of *open boundary ASEP polynomials*, which are closely related to Koornwinder polynomials. In [7], I introduce the *stoned multispecies open boundary ASEP* and show that its stationary distribution is given by open boundary ASEP polynomials evaluated at generic values. In a special case, the stoned multispecies open boundary ASEP can be interpreted as a random combinatorial billiard trajectory in a torus obtained from the affine Weyl group $\widetilde{C}_n$.

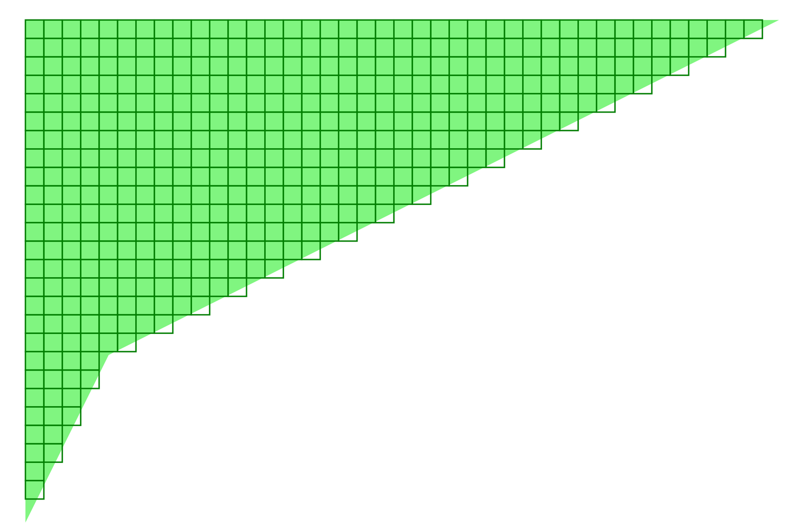


Core Partitions

Let $W = \mathfrak{S}_n$. There is a natural correspondence between alcoves in the fundamental chamber of the braid arrangement and n-core partitions. Thus, after quotienting by the action of $\mathfrak{S}_n$, one can view the reduced random billiard trajectory as a random growth model for n-core partitions. In [7], I compute the scaling limit. As $n \to \infty$, the scaling limits converge to the region

$$\left\{(x,y) \in \mathbb{R}^2 : y \le 0 \le x,\ \sqrt{(1-p)x} + \sqrt{-y} \le (6(1-p))^{1/4}\right\}.$$





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