Geometric realizations of ν-associahedra via brick polyhedra

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Coxeter Groups (Type A_n)

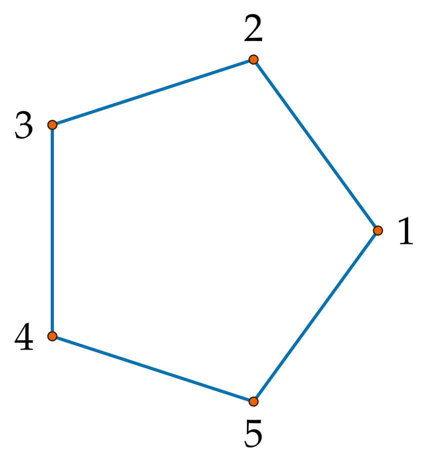
We denote the i^{th} unit vector in $\mathbb{R}^n$ by ϵ_i and define for $i \in [n]$:

- **Simple roots:** $\alpha_i := \epsilon_i - \epsilon_{i+1}$
- **Fundamental weights:** $\omega_i := \sum_{j \le i} \epsilon_j$
- **Generators:** reflections along simple roots $s_i := s_{\alpha_i}$

Coxeter group W_{A_n}: symmetric group $\mathcal{S}_{n+1}$ generated by $S = \{s_1, \dots, s_n\}$

Subword Complexes $\mathcal{SC}(Q, w)$ [5]

Subword complex $\mathcal{SC}(Q, w)$: For a word $Q = (q_1, \dots, q_n)$ in S and $w \in W_{A_n}$, simplicial complex whose facets are $I \subseteq [n]$, such that $Q_{[n]\setminus I}$ is a reduced expression of w.



Subword complex for $Q = (s_1, s_2, s_1, s_2, s_1)$, $w = s_1 s_2 s_1$.

Brick Polyhedra $\mathcal{B}(Q, w)$ [4]

For I facet of $\mathcal{SC}(Q, w)$, $k \in [n]$:

- **root function:** $\mathrm{r}(I, k) := \prod Q_{\{1,\dots,k-1\}\setminus I}(\alpha_{q_k})$
- **root configuration:** $\mathrm{R}(I) = \{\mathrm{r}(I, k) | 1 \le k \le n\}$
- **weight function:** $\mathrm{w}(I, k) := \prod Q_{\{1,\dots,k-1\}\setminus I}(\omega_{q_k})$
- **brick vector:** $b(I) := -\sum_{k=1}^{n} \mathrm{w}(I, k)$
- **upper Bruhat cone:** $\mathcal{C}^+(w, \mathrm{Dem}(Q)) = \bigcap_{I \text{ facet}} \mathrm{cone}\ \mathrm{R}(I)$
- **brick polyhedron:** $\mathcal{B}(Q, w) := \mathrm{conv}\{b(I) \mid I \text{ facet of } \mathcal{SC}(Q, w)\} + \mathcal{C}^+(w, \mathrm{Dem}(Q))$
- The brick polyhedron satisfies the **local cone property**: $\mathrm{cone}^{(b(I))}(\mathcal{B}(Q, w)) = \mathrm{cone}\ \mathrm{R}(I)$.

Compatibility and ν-Trees

- **northeast path ν:** lattice path using north and east steps of unit length
- **ν-incompatible nodes:** lattice points inside Ferrers diagram such that

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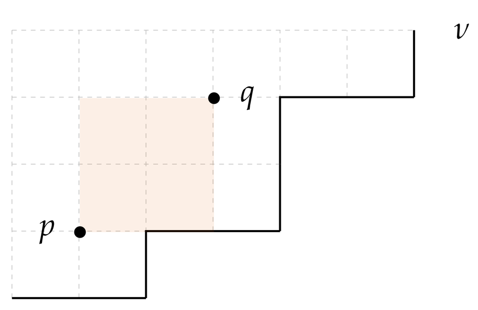
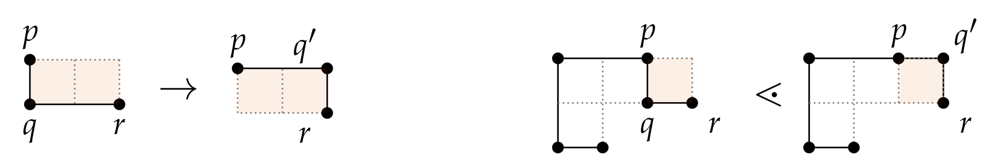


- **ν-tree:** maximal set of ν-compatible points

ν-Tamari Lattice [3] [6]

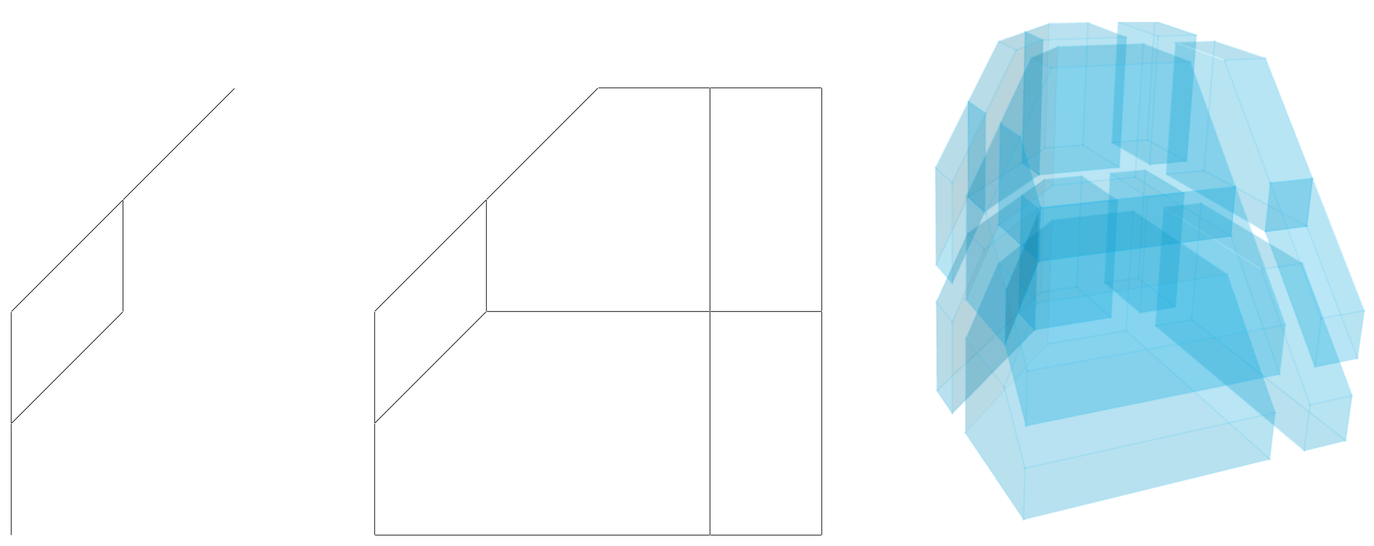
- **rotation:** change node q by q'



- **ν-Tamari lattice:** ν-trees ordered by rotation
- **ν-Tamari complex:** simplicial complex of pairwise ν-compatible sets of points

The ν-Associahedron [2]

- **ν-associahedron:** polytopal complex dual to complex of interior faces of the ν-Tamari complex
  - vertices: ν-trees
  - edge graph: Hasse diagram of the ν-Tamari lattice



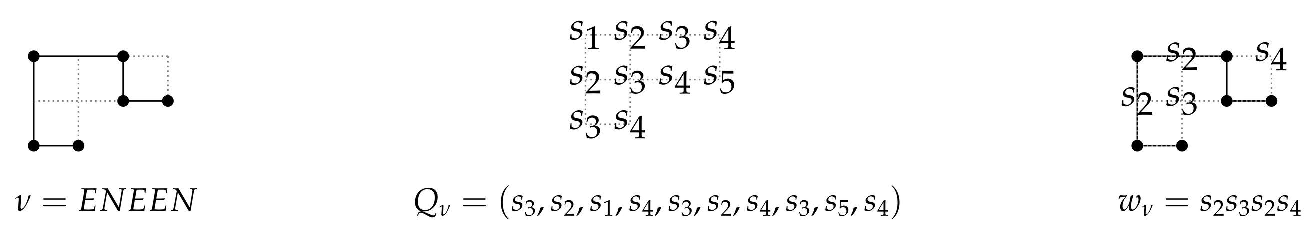
$\nu = EENN$ $\nu = EENEEN$ $\nu = EENEENEEN$

Goal of this presentation

Present a geometric realization of the ν-associahedron.

The ν-Subword Complex $\mathcal{SC}(Q_\nu, w_\nu)$ [3]

- $d(p)$: lattice distance from p to the top-left corner
- **labeling:** label each lattice point of the Ferrers diagram by the transposition $s_{d(p)+1}$
- Q_ν: read transpositions: bottom to top, left to right
- w_ν: read transpositions of the complement of ν-tree

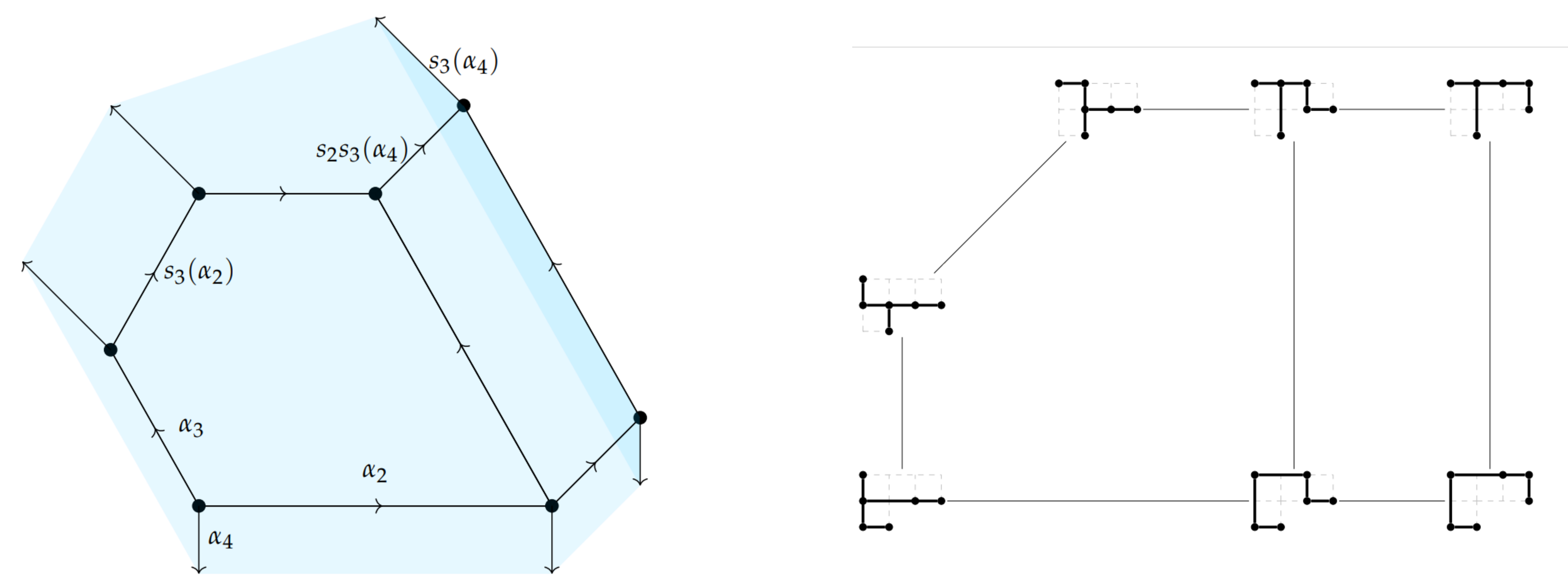


$\nu = ENEEN$ $Q_\nu = (s_3, s_2, s_1, s_4, s_3, s_2, s_4, s_3, s_5, s_4)$ $w_\nu = s_2 s_3 s_2 s_4$

- ν-subword complex $\mathcal{SC}(Q_\nu, w_\nu)$ is isomorphic to the ν-Tamari complex.

ν-Brick Polyhedron $\mathcal{B}(Q_\nu, w_\nu)$

- **ν-brick polyhedron $\mathcal{B}(Q_\nu, w_\nu)$:** brick polyhedron of ν-subword complex $\mathcal{SC}(Q_\nu, w_\nu)$



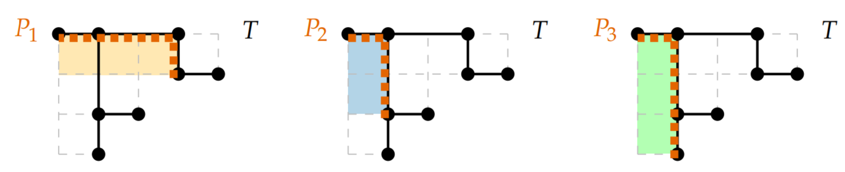
Comparison of the ν-brick polyhedron and ν-associahedron for $\nu = ENEEN$.

Main Therem [Ceballos - Müller, 2025]

The bounded faces of the ν-brick polyhedron $\mathcal{B}(Q_\nu, w_\nu)$ give a geometric realization of the ν - associahedron.

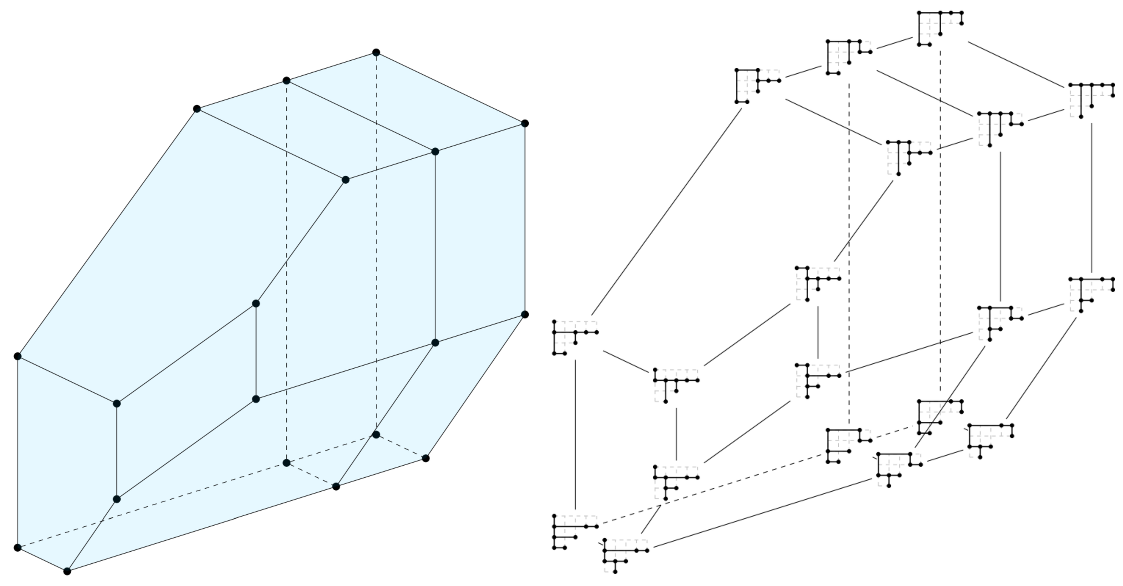
A Projection - Special Case

- **canonical coordinates** $y(T)$: The entry $y_i(T)$ is the area (i.e. number of boxes to the left) of the path $P_i(T)$ connecting the root to the leftmost node of T at level i (increasing from top to bottom).



Canonical coordinates: $y(T) = (3, 2, 3)$.

- **special case:** No consecutive north steps: projected points coincide with Ceballos's canonical realization [1].



Left: Projection of the bounded components, Right: ν-associahedron.

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