# Differential transcendence and walks on self-similar graphs

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## Context

We can sort series in combinatorics by the equations they satisfy. They might be rational, algebraic, D-finite, differentially algebraic, or none of these. Combinatorial complexity of walks on graphs is often mirrored by these categories.

## Random walks on infinite graphs

For a connected graph $X$, define for each directed edge $(x, y)$ the transition probability $\mathbb{P}(x,y) = \frac{1}{\deg(x)}$. The probability of a walk is the product of each edge probability. The **Green's function** is a sum over walks $\omega$ from $x$ to $y$:

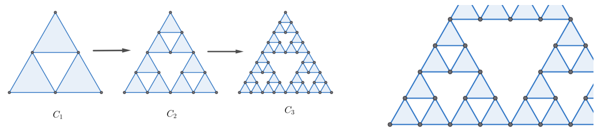
$$G(x,y|z) := \sum_{\omega: x\to y} \mathbb{P}(\omega) z^{|\omega|}$$

The graphs here will have a fixed origin vertex $o$. We focus on $G(z) := G(o,o|z)$.
**For which graphs is $G(z)$ algebraic? Can we combinatorially characterize them?**

## Symmetrically self-similar graphs

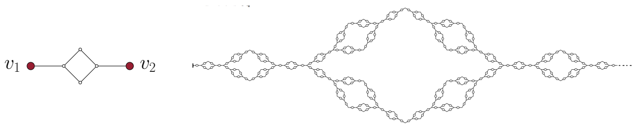
Symmetrically self-similar graphs are a well studied class of fractal graphs, constructed as follows:

1. Start with a **cell graph** $C_1$. It has certain symmetry properties, and is built by connecting some complete graphs of the same order, $\theta$, the **branching number**.
2. Next, iterate a **blowing** operation which creates $C_n$ by replacing each clique in $C_{n-1}$ by $C_1$.
3. The limit, $C_\infty$, is a symmetrically self-similar graph with a unique origin vertex $o$, the image of $v_1$ in $C_1$.




(a) The iterative process giving the Sierpiński triangle. $\theta = 3$

(b) The Sierpiński triangle $C_\infty$




(a) A cell graph $C_1$ with $\theta = 2$

(b) The associated symmetrically self-similar graph $C_\infty$

For this graph $C_\infty$,

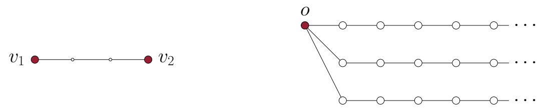
$$G_{C_\infty}(z) = 1 + \frac{z^2}{3} + \frac{2z^4}{9} + \frac{5z^6}{27} + O(z^8)$$

**Remark:** Gluing multiple copies of $C_\infty$ at the origin is also a symmetrically self-similar graph, with the same main Green's function.

Long version: https://arxiv.org/abs/2411.19316

## A star graph

A path cell graph leads to a star graph. Star graphs are the only known family of symmetrically self-similar graphs with an algebraic Green's function.




(a) A path ($\theta = 2$) (b) We join multiple copies of the limit (an infinite path) at the origin

The Green's function is algebraic:

$$G(z) = \sum_n \frac{\binom{2n}{n}}{2^{2n}} z^{2n} = \frac{1}{\sqrt{1-z^2}}$$

## Our main result

THEOREM. Let $X$ be a symmetrically self-similar graph with bounded geometry, origin $o$, and branching number $\theta = 2$.

**Either**

The graph is a **star** consisting of finitely many one-sided lines that coincide at $o$.

**or**

The Green's function $G(o, o|z)$ of $X$ is **differentially transcendental** over $\mathbb{C}(z)$.

## Proof Ingredients

**Grabiner and Woess:**. Green's functions of symmetrically self-similar graphs satisfy functional equation

$$G(z) = f(z)G(d(z)).$$

**by Di Vizio et al.:**. A solution to the functional equation is either algebraic, or it is differentially transcendental. If it is algebraic, there is an $N$ so $G^N = P(z)/Q(z)$.

**by Grabiner and Woess:**. It $G$ is algebraic the singular expansion around the dominant singularity is $(1-z)^\eta$, and the singularities are limited to $(-\infty, -1)$.

**Key step:**. If $G$ is algebraic, $\eta \le \frac{1}{2}$ with equality if and only if $X$ is the star graph.

## Next steps

**Conjecture:** The result is also true for all $\theta > 2$.

**Infinite cell graphs** Some self-similar graphs generated from **infinite** cell graphs are Cayley graphs with algebraic Green's function. **Is there a wider theory?**

**Conjecture** Teufl conjectured that the Julia set of the poles of the Green's function of self-similar graphs is a Cantor set, except for star graphs. We can partially establish this result as a consequence of our Main Result.

## Definitions

A cell graph $C_1$ with branching number $\theta$, is finite and made from $\mu$ complete graphs and have $\theta$ extremal points. It must have certain automorphisms switching those extremal points.

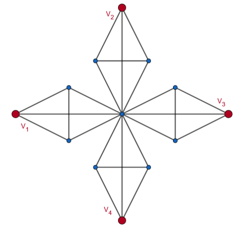



Figure: A cell graph with $\theta = 4$. The 4 extremal vertices are in red.

- A series in $\mathbb{C}[[z]]$ is **differentially algebraic** over $\mathbb{C}(z)$ if it satisfies a polynomial differential equation over $\mathbb{C}(z)$. Otherwise, it is **differentially transcendental**.
- $f(z)$ : generating function of probabilities of walks on the cell starting and ending at $v_1$, never touching $v_2, \ldots, v_\theta$.
- $d(z)$ : generating function of probabilities of walks on the cell starting at $v_1$ ending at one of $v_2, \ldots, v_\theta$.
- $\tau = d'(1)$ is the expected number of steps for a random walk starting at $v_1$ for reaching $v_2, \ldots, v_\theta$
- $\eta = \frac{\log \mu}{\log \tau} - 1$ is the critical exponent of the asymptotics of the Green's function at the dominant singularity.

## References


Lucia Di Vizio, Gwladys Fernandes, and Marni Mishna. Inhomogeneous order 1 iterative functional equations with applications to combinatorics. *arXiv preprint arXiv:2309.07680*, 2023.

Peter J. Grabner and Wolfgang Woess. Functional iterations and periodic oscillations for simple random walk on the Sierpiński graph. *Stochastic Processes and their Applications*, 69(1):127–138, 1997. ISSN 0304-4149. doi: 10.1016/S0304-4149(97)00033-1. URL https://doi.org/10.1016/S0304-4149(97)00033-1.

Bernhard Krön. Green functions on self-similar graphs and bounds for the spectrum of the Laplacian. *Université de Grenoble. Annales de l'Institut Fourier*, 52(6):1875–1900, 2002. ISSN 0373-0956,1777-5310. URL http://aif.cedram.org/item?id=AIF_2002__52_6_1875_0.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