
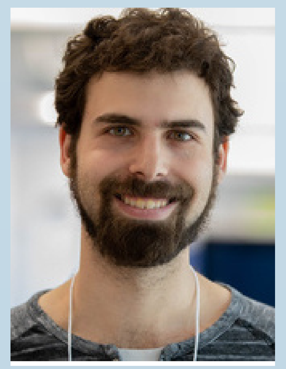
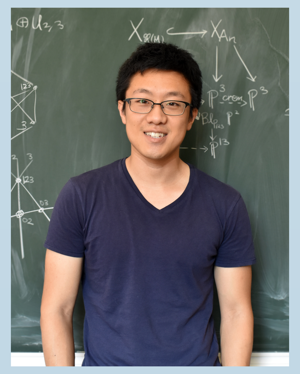

ON TWO NOTIONS OF TOTAL POSITIVITY FOR GENERALIZED PARTIAL FLAG VARIETIES OF CLASSICAL LIE TYPES

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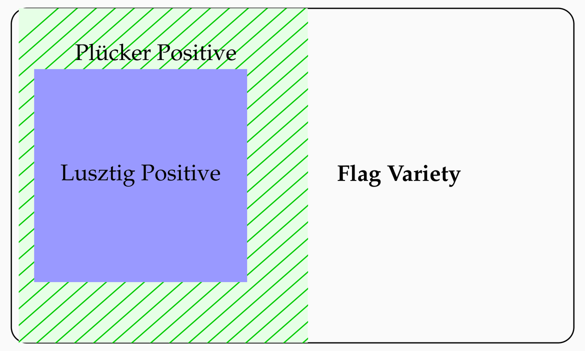
Outline

- Bloch and Karp [BK] showed that two natural notions of nonnegativity coincide for many partial flag varieties of type A.
- Type B or C partial flag varieties can be realized as a subset of type A partial flag varieties satisfying some extra conditions.
- There are two analogous notions of nonnegativity for type B and C partial flag varieties.
- **Key Idea:** For each definition of nonnegativity, we show that the nonnegative part of the type B or C flag variety lies within the corresponding nonnegative type A flag variety.
- We conclude that in many cases, the two notions coincide.



Flag Varieties

(A) The rank $\mathbf{r} = (r_1, \dots, r_k)$ *Type A Flag Variety* in $\mathbb{R}^n$ is:

$$\mathrm{Fl}_{\mathbf{r};n} = \{L_1 \subset \cdots \subset L_k \subset \mathbb{R}^n \mid \dim(L_i) = r_i \text{ for } i \in [k]\}.$$

(B) Let E^B be the $(2n+1) \times (2n+1)$ symmetric matrix $\begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \iddots & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}$.

The **Type B Flag Variety** of rank $\mathbf{r}$ is:

$$\mathrm{SOFl}_{\mathbf{r};2n+1} = \left\{(L_1, \dots, L_k) \in \mathrm{Fl}_{\mathbf{r};n} \mid vE^B w^t = 0 \ \ \forall\, v, w \in L_k\right\}.$$

(C) Let E^C be the $(2n) \times (2n)$ skew-symmetric matrix $\begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \iddots & & \\ & 1 & & & \\ -1 & & & & \end{pmatrix}$.

The **Type C Flag Variety** of rank $\mathbf{r}$ is:

$$\mathrm{SpFl}_{\mathbf{r};2n} = \left\{(L_1, \dots, L_k) \in \mathrm{Fl}_{\mathbf{r};n} \mid vE^C w^t = 0 \ \ \forall\, v, w \in L_k\right\}.$$

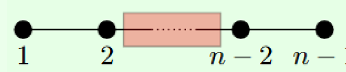
Notation: We will use $\bigstar\mathrm{Fl}_{\mathbf{r};n}$ *when we wish to denote any of* $\mathrm{Fl}_{\mathbf{r};n}$, $\mathrm{SOFl}_{\mathbf{r};2n+1}$, *or* $\mathrm{SpFl}_{\mathbf{r};2n}$.

Type A result

Our goal is a type B and C analogue of:

Theorem ([BK]). *The following are equivalent:*

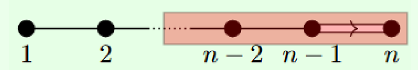
1. *The Plücker positive and Lusztig positive type A flag varieties coincide:*
$$\mathrm{Fl}^{>0}_{\mathbf{r};n} = \mathrm{Fl}^{\Delta>0}_{\mathbf{r};n}$$
2. *The Plücker non-negative and Lusztig non-negative type A flag varieties coincide:*
$$\mathrm{Fl}^{\geq 0}_{\mathbf{r};n} = \mathrm{Fl}^{\Delta\geq 0}_{\mathbf{r};n}$$
3. *The rank vector consists of consecutive integers:*
$$\mathbf{r} = (a, a+1, \dots, b)$$

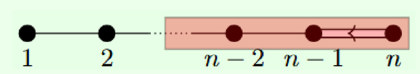


Main result

Theorem ([BBEG]). *Let* $n \geq 3$. *Then the following are equivalent:*

1. $\mathrm{SOFl}^{>0}_{\mathbf{r};2n+1} = \mathrm{SOFl}^{\Delta>0}_{\mathbf{r};2n+1}$
2. $\mathrm{SOFl}^{\geq 0}_{\mathbf{r};2n+1} = \mathrm{SOFl}^{\Delta\geq 0}_{\mathbf{r};2n+1}$
3. $\mathrm{SpFl}^{>0}_{\mathbf{r};2n} = \mathrm{SpFl}^{\Delta>0}_{\mathbf{r};2n}$
4. $\mathrm{SpFl}^{\geq 0}_{\mathbf{r};2n} = \mathrm{SpFl}^{\Delta\geq 0}_{\mathbf{r};2n}$
5. $\mathbf{r} = (a, a+1, \dots, n)$





For $n = 2$, items 1 and 2 are no longer equivalent to the rest.

Plücker Nonegativity

We represent flags in $\bigstar\mathrm{Fl}_{\mathbf{r};n}$ by $n \times n$ matrices (non-unqiuely).

$$M \quad \longleftrightarrow \quad \begin{array}{c} F = (L_1 \subset \cdots \subset L_k) \\ L_i = \mathrm{rowspan}(\text{first } r_i \text{ rows of } M) \end{array}$$

The Plücker coordinates of F are certain minors of M. Specifically, for each $i \in [k]$, take all the $r_i \times r_i$ minors of M in rows $1, \dots, r_i$.

- The **Plücker positive flag variety** is:
$$\bigstar\mathrm{Fl}^{\Delta>0}_{\mathbf{r};n} = \left\{F \in \bigstar\mathrm{Fl}_{\mathbf{r};n} \mid \text{All Plücker coordinates are } > 0\right\}.$$
- The **Plücker nonnegative flag variety** is:
$$\bigstar\mathrm{Fl}^{\Delta\geq 0}_{\mathbf{r};n} = \left\{F \in \bigstar\mathrm{Fl}_{\mathbf{r};n} \mid \text{All Plücker coordinates are } \geq 0\right\}.$$

Lusztig nonnegativity

We describe a parameterization of the Lusztig positive part of $\bigstar\mathrm{Fl}_{\mathbf{r};n}$ using the following matrices, which come from a choice of **Chevalley generators** for the Lie algebra of the appropriate type. Making this choice corresponds to choosing a **pinning** of the linear algebraic group of the appropriate type.

Let $E_{i,j}$ be the matrix with a 1 in row i, column j and zeros elsewhere.

(A) $y^A_i(t) = I + tE_{i,i+1}$ for $i \in [n-1]$

(B) $y^B_i(t) = I + tE_{i,i+1} + tE_{2n+1-i,2n+2-i}$ for $i \in [n-1]$.
$y^B_n(t) = I + \sqrt{2}tE_{n,n+1} + \sqrt{2}tE_{n+1,n+2} + t^2E_{n,n+2}$

(C) $y^C_i(t) = I + tE_{i,i+1} + tE_{2n-i,2n-i+1}$ for $i \in [n-1]$.
$y^C_n(t) = I + tE_{n,n+1}$

For example, when $n = 2$, $y^B_1(t) = \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and $y^C_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

The following equations hold:

$$y^B_i(t) = y^A_i(t)y^A_{2n+1-i}(t), \quad i \in [n-1] \qquad y^C_i(t) = y^A_i(t)y^A_{2n-i}(t), \quad i \in [n-1]$$
$$y^B_n(t) = y^A_n(\tfrac{t}{\sqrt{2}})y^A_{n+1}(\sqrt{2}t)y^A_n(\tfrac{t}{\sqrt{2}}) \qquad y^C_n(t) = y^A_n(t)$$

Recall that $\bigstar$ denotes type A, B, or C. Let $\mathbf{i}$ be an expression for the longest word in $(W^\bigstar)^J$, where $W^\bigstar$ is the type $\bigstar$ Weyl group and J is the complement of $\mathbf{r}$ in the roots of the appropriate type. [*Alternatively, one may take* $\mathbf{i}$ *to be any sufficiently long random sequence in* $[n-1]$ *(type A) or in* $[n]$ *(types B or C)*].

Let $\mathbf{i} = (i_1, \dots, i_N)$. The **Lusztig positive flag variety**, due to Lusztig [Lus], is:

$$\bigstar\mathrm{Fl}^{>0}_{\mathbf{r};n} = \left\{\prod_{j=1}^{N} y_{i_j}(t_j) \;\middle|\; \forall\, j \in [N],\ t_j > 0\right\}.$$

The **Lusztig nonnegative flag variety** is:

$$\bigstar\mathrm{Fl}^{\geq 0}_{\mathbf{r};n} = \overline{\bigstar\mathrm{Fl}^{>0}_{\mathbf{r};n}}.$$

Bibliography

[BBEG] G. Barkley, J. Boretsky, C. Eur, and J. Gao. "On two notions of total positivity for generalized partial flag varieties of classical Lie types". 2024. arXiv:2410.11804

[BK] A. M. Bloch and S. N. Karp. "On two notions of total positivity for partial flag varieties". Adv. Math. 414 (2023)

[Lus] G. Lusztig. "Total positivity in partial flag manifolds". Represent. Theory 2 (1998), pp. 70–78.

Example

Let $n = 2$ and $\mathbf{r} = (1,2)$. We give an example of a flag in $\mathrm{SpFl}^{>0}_{(1,2);2} = \mathrm{SpFl}^{\Delta>0}_{(1,2);2}$. We start with an expression for the longest word in the type C Weyl group, $s_1s_2s_1s_2$. For positive parameters a, b, c, d, construct a Lusztig positive flag represented by:

$$y^C_1(a)y^C_2(b)y^C_1(c)y^C_2(d) = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & d & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+c & ab+ad+cd & abc \\ 0 & 1 & b+d & bc \\ 0 & 0 & 1 & a+c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can observe that this matrix represents a flag in $SpFl_{(1,2);3}$. For instance, applying the form E^C to the first 2 rows gives

$$(1)(bc) - (a+c)(b+d) + (ab+ad+cd)(1) - (abc)(0) = 0.$$

Observe that this is also Plücker positive. For instance,

$$P_2 = a+c,\ P_4 = abc,\ P_{23} = bc,\ P_{24} = bc^2, \text{ and } P_{34} = bc^2d.$$

Conversely, observe that for matrices of this form,

$$a = \frac{P_4}{P_{23}},\ b = \frac{(P_{23})^2}{P_{24}},\ c = \frac{P_{24}}{P_{23}}, \text{ and } d = \frac{P_{34}}{P_{24}}.$$

Moreover, one can show that the Plückers in these expressions uniquely determine all others. Thus, Plücker positivity implies Lusztig positivity.

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