A permutation based approach to the q-deformation of the Dynkin Operator

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Integer compositions

Comp_n is the set of compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ of integer n. For a set $I = \{i_1 < \cdots < i_p\} \subseteq [n-1]$, let $\mathrm{comp}(I) = (i_1, i_2 - i_1, \ldots, n - i_p) \in \mathrm{Comp}_n$.
We say that $\alpha = (\alpha_1, \ldots, \alpha_m)$ *refines* $\beta = (\beta_1, \ldots, \beta_p)$, and write $\alpha \preceq \beta$, iff α can be split into p compositions $\alpha^i = (\alpha^i_1, \ldots, \alpha^i_{m_i})$ with $\alpha^i \in \mathrm{Comp}_{\beta_i}$. Denote by $\alpha|\beta$ this sequence, and by $\overline{\alpha|\beta}$ the list of the rightmost elements of each α^i, that is,

$$\overline{\alpha|\beta} = (\alpha^1_{m_1}, \alpha^2_{m_2}, \ldots, \alpha^p_{m_p}).$$

For I and J subsets of $[n-1]$, write $I|J$ (resp. $\overline{I|J}$) instead of $\mathrm{comp}(I)|\mathrm{comp}(J)$ (resp. $\overline{\mathrm{comp}(I)|\mathrm{comp}(J)}$).
For U sequence of numbers, we let $|U|$ (resp. ΣU) denote the number (resp. the sum) of its elements, i.e., we set $\Sigma U := \sum_{u \in U} u$.

Solomon's descent algebra

Given a permutation $\pi \in S_n$, we consider its *descent set* $\mathrm{Des}(\pi)$ defined as $\mathrm{Des}(\pi) = \{1 \leq i \leq n-1 | \pi(i) > \pi(i+1)\}$. For $I \subseteq [n-1]$, let $\mathbf{D}_I$ and $\mathbf{B}_I \in \mathbf{k}S_n$ defined as

$$\mathbf{D}_I = \sum_{\pi \in S_n;\ \mathrm{Des}(\pi) = I} \pi, \qquad \mathbf{B}_I = \sum_{\pi \in S_n;\ \mathrm{Des}(\pi) \subseteq I} \pi.$$

They are two bases of the *descent algebra* Σ_n, a subalgebra of $\mathbf{k}S_n$ of dim. 2^{n-1}.

Prop (Solomon's Mackey formula). *Let $I, J, K \subseteq [n-1]$. Let $\mathbb{N}^{I,J}_K$ be the set of nonnegative integer matrices with row sums vector $\mathrm{comp}(I)$, column sums vector $\mathrm{comp}(J)$ and reduced reading word $\mathrm{comp}(K)$. Then, $[\mathbf{B}_K]\mathbf{B}_J\mathbf{B}_I = |\mathbb{N}^{I,J}_K|$.*

Thus $[\mathbf{B}_K]\mathbf{B}_J\mathbf{B}_I = 0$ unless $I \subseteq K$, and the non-zero entries of the i-th row in the matrix must give the i-th subsequence in $K|I$. Furthermore the sequence of the rightmost non-zero elements of each row in the matrix is exactly $\overline{K|I}$.

Exm. Assume $n = 5$, $I = \{3\}$, $J = K = \{1, 3\}$. We have $\mathrm{comp}(I) = (3, 2)$, $\mathrm{comp}(J) = \mathrm{comp}(K) = (1, 2, 2)$. We notice that $I \subseteq K$, thus $\mathrm{comp}(K) \preceq \mathrm{comp}(I)$. Finally $K|I = ((1, 2), (2))$, $\overline{K|I} = (2, 2)$ and there are $|\mathbb{N}^{I,J}_K| = 2$ suitable matrices:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The Dynkin operator

Def (Dynkin operator). The *Dynkin operator* is the element

$$\mathbf{V}_n = \sum_{k=1}^{n} (-1)^{k-1} \mathbf{D}_{[k-1]} \in \Sigma_n.$$

It satisfies $\mathbf{V}_n^2 = n\mathbf{V}_n$. This is proved in [2] using the free Lie algebra, and in [3] using custom-built combinatorics to show that $\mathbf{V}_n\mathbf{B}_I = 0$ if $I \subseteq [n-1]$, $I \neq \emptyset$. This suggests a link to Solomon's Mackey formula.

Def (q-deformation of the Dynkin operator, [4]). Fix $q \in \mathbf{k}$ and non-negative integer n.

$$\mathbf{V}^{(q)}_n := \sum_{k=1}^{n} (-q)^{k-1} \mathbf{D}_{[k-1]} \in \mathbf{k}S_n. \qquad (1)$$

Action of the q-Dynkin operator on the B-basis

Thm. *Let n be a positive integer and $q \in \mathbf{k}$ be an invertible parameter. For $I, K \subseteq [n-1]$, the coefficient in $\mathbf{B}_K$ of $\mathbf{V}^{(q)}_n\mathbf{B}_I$ is 0 if $I \not\subseteq K$ and otherwise given by*

$$[\mathbf{B}_K]\mathbf{V}^{(q)}_n\mathbf{B}_I = (-1)^{|K|} q^{n-1} (1 - q^{-1})^{|I|} \prod_{v \in \overline{K|I}} [v]_{q^{-1}}. \qquad (2)$$

Proof. Define $\mathbf{V}_{n,k} := \mathbf{D}_{[k-1]}$ for each $k \in [n]$. Given two positive integers $k \leq n$ and two sets $I, K \subseteq [n-1]$ with $I \subseteq K$, define $\mathbb{N}^I_K := \bigcup_{J \subseteq [n-1]} \mathbb{N}^{I,J}_K$. Given $A \in \mathbb{N}^I_K$, we let $\mathrm{col}(A)$ be the number of columns of A and $A^\star$ the sequence of entries in the rightmost column of A. Use the Solomon's Mackey formula to show:

$$[\mathbf{B}_K]\mathbf{V}_{n,k}\mathbf{B}_I = \sum_{J \subseteq [k-1]} (-1)^{(k-1)-|J|} [\mathbf{B}_K]\mathbf{B}_J\mathbf{B}_I = \sum_{\substack{A \in \mathbb{N}^I_K; \\ \Sigma A^\star > n-k}} (-1)^{k - \mathrm{col}(A)}. \qquad (3)$$

We establish a sign-reversing involution on the set $\mathbb{N}^I_K$ that makes most addends on the right hand side of the above equation cancel. Call a column of a matrix $A \in \mathbb{N}^I_K$ *splittable* if it is not the last column and has at least two nonzero entries. Conversely, two adjacent columns c' and c of A (with c' left of c) are called *mergeable* if c is not the last column of A and the column c' has only one nonzero entry and this entry lies further up than all nonzero entries of c. Splitting and merging are mutually inverse operations.

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix} \xrightarrow{\text{split column 2}} \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 2 \end{pmatrix} \xrightarrow{\text{merge columns 2,3}} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

Thus, all matrices $A \in \mathbb{N}^I_K$ that have a splittable column or a mergeable pair of columns cancel out from (3). What remains are the *survivor matrices* A of the form:

$$A = \begin{pmatrix} & & & & p \\ & & & w & q \\ & & & & r \\ & y & z & & s \\ x & & & & t \end{pmatrix}$$

A survivor matrix $A \in \mathbb{N}^I_K$ is **entirely determined** by the knowledge of which of its rows end with a 0. If the i-th row of the matrix does not end with a zero, then it ends precisely with the i-th element of $\overline{K|I}$. We can therefore encode this data as a nonempty subsequence U of $\overline{K|I}$, which consists of those entries of $\overline{K|I}$ that are in the last column of A. There are exactly $|\mathrm{comp}(K)| - |U| = |K| + 1 - |U|$ non rightmost columns in A. Add the last column to get $\mathrm{col}(A) = 2 + |K| - |U|$.

Exm. Let $\mathrm{comp}(K) = (1, 2, 3, 1)$ and $\mathrm{comp}(I) = (3, 4)$ so that $\overline{K|I} = (2, 1)$. Then, the survivor matrices A are

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 3 & 1 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}.$$

Their respective subsequences U are $U_1 = (2)$, $U_2 = (1)$ and $U_3 = (2, 1)$.

As a consequence, the $\mathbf{B}_K$-coefficient in the expansion in the $\mathbf{B}$-basis of $\mathbf{V}_{n,k}\mathbf{B}_I$ is

$$[\mathbf{B}_K]\mathbf{V}_{n,k}\mathbf{B}_I = (-1)^{k+|K|} \sum_{\substack{U \subseteq \overline{K|I}; \\ \Sigma U > n-k}} (-1)^{|U|}. \qquad (4)$$

Summing (4) over k yields our main result. □

Idempotence, zero coefficients and eigenvalues

Cor. The facts that $\mathbf{V}_n^2 = n\mathbf{V}_n$ and $\mathbf{V}_n\mathbf{B}_I = 0$ if $I \subseteq [n-1]$, $I \neq \emptyset$ are direct consequences of our main theorem.

Cor. Let $\mathbf{k}$ be a field of characteristic 0. Fix $n \in \mathbb{N}$ and $q \in \mathbf{k}$ and $I, K \subseteq [n-1]$.

- If q is a primitive p-th root of unity for $p > 1$, then we have $[\mathbf{B}_K]\mathbf{V}^{(q)}_n\mathbf{B}_I = 0$ iff $I \not\subseteq K$ or $\overline{K|I}$ contains a multiple of p.
- If $q = 1$, then $[\mathbf{B}_K]\mathbf{V}^{(q)}_n\mathbf{B}_I = 0$ iff $I \neq \varnothing$.
- If q is nonzero and not a root of unity, then $[\mathbf{B}_K]\mathbf{V}^{(q)}_n\mathbf{B}_I = 0$ iff $I \not\subseteq K$.

Cor. Fix $n > 0$ and $q \neq 1$. The eigenvalues of $\mathbf{V}^{(q)}_n$ are the $(e_I)_{I \subseteq [n-1]}$ defined by

$$(1 - q)e_I = \prod_{v \in \mathrm{comp}(I)} (1 - q^v). \qquad (5)$$

If q is a p-th root of unity with $p > 1$ and I contains a multiple of p, then $e_I = 0$.

Image space dimension

We proceed with a corollary and some comments regarding the dimension of $\mathbf{V}^{(q)}_n\Sigma_n$.

Cor. Let $n > 0$. Let $\mathbf{k}$ be a field. The dimension of the subspace $\mathbf{V}^{(q)}_n\Sigma_n$ of the descent algebra Σ_n is 2^{n-1} when q is not a root of unity. If q is a primitive p-th root of unity with $p > 1$,

$$\dim\left(\mathbf{V}^{(q)}_n\Sigma_n\right) = s^{(p)}_n, \qquad (6)$$

where $s^{(p)}_n$ is the n-th Fibonacci number of order p, $s^{(p)}_n = s^{(p)}_{n-1} + s^{(p)}_{n-2} + \cdots + s^{(p)}_{n-p}$.

1. For $p = 2$ i.e. $q = -1$, the basis of the image of $\mathbf{V}^+_n := \mathbf{V}^{(-1)}_n$ is indexed by *odd compositions* of n which reminds one of the *peak algebra* P_n of order n and $\mathbf{V}^+_n\mathbf{B}_I = 2^{|I|}\sum_{K \supseteq I; \overline{K|I} \cap 2\mathbb{N} = \emptyset} (-1)^{n-1-|K|}\mathbf{B}_K$.

2. When q is a primitive p-th root of unity, the dimension of the image of $\mathbf{V}^{(q)}_n$ is equal to the dimension of the vector space of $(p-1)$*-extended peaks quasisymmetric functions of degree* n that we introduced in [7].

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