The monopole-dimer model on high-dimensional cylindrical, toroidal, Möbius and Klein grids

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Objective

To generalize the monopole-dimer model for high-dimensional grid graphs with different boundary conditions.

Loop-vertex Model [1]

The loop-vertex model on the (edge- and vertex-weighted) graph G with an orientation $\mathcal{O}$ is the collection $\mathcal{L}$ of configurations consisting directed even loops, doubled edges and some isolated vertices with the weight of each configuration C defined as:

$$w(C) = \prod_{\ell = \text{loop in } C} w(\ell) \prod_{\substack{v \text{ an} \\ \text{isolated vertex} \\ \text{in } C}} x(v)$$

$$\text{where } w(\ell) = -\prod_{i=1}^{2m} sgn(v_i, v_{i+1})\, a_{v_i, v_{i+1}} \quad \text{for } \ell = (v_1, v_2, .., v_{2m}, v_1).$$

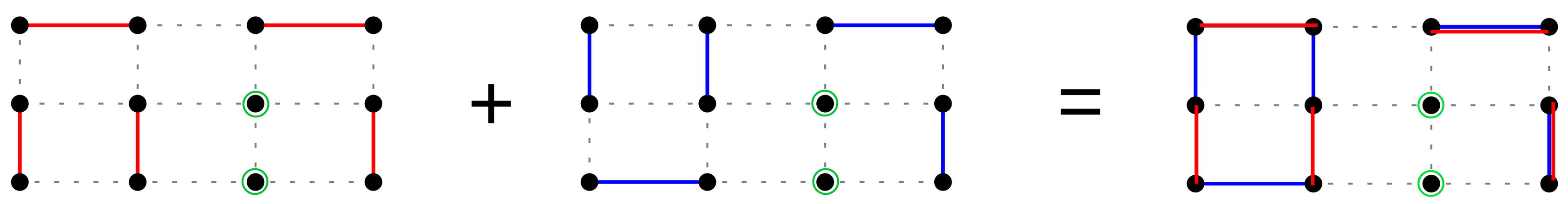

Figure: Two matchings overlapping to form a loop-vertex configuration.

The partition function of the loop-vertex Model on $(G, \mathcal{O})$ is defined as

$$\mathcal{Z}_{G,\mathcal{O}} := \sum_{C \in \mathcal{L}} w(C).$$

Partition function is a determinant [1]

The partition function of the loop-vertex model on $(G, \mathcal{O})$ is

$$\mathcal{Z}_{G,\mathcal{O}} = \det(\mathcal{K}_{G,\mathcal{O}}),$$

where $\mathcal{K}_{G,\mathcal{O}}$ is a generalised adjacency matrix of $(G, \mathcal{O})$ defined as:

$$\mathcal{K}_{G,\mathcal{O}}(v, v') = \begin{cases} x(v) & \text{if } v = v', \\ a_{v,v'} & \text{if } v \to v' \text{ in } \mathcal{O}, \\ -a_{v,v'} & \text{if } v' \to v \text{ in } \mathcal{O}, \\ 0 & \text{if } (v, v') \notin E(G). \end{cases} \quad (1)$$

Oriented Cartesian product

The *oriented Cartesian product* of naturally labeled oriented graphs $(G_1, \mathcal{O}_1), \dots, (G_k, \mathcal{O}_k)$ is the graph $G_1 \square \cdots \square G_k$ with orientation $\mathcal{O}$ given as follows. For each $i \in [k]$, if $u_i \to u_i'$ in $\mathcal{O}_i$, then $\mathcal{O}$ gives orientation $(u_1, \dots, u_i, \dots, u_k) \to (u_1, \dots, u_i', \dots, u_k)$ if $u_{i+1} + u_{i+2} + \cdots + u_k + (k - i) \equiv 0 \pmod 2$ and $(u_1, \dots, u_i', \dots, u_k) \to (u_1, \dots, u_i, \dots, u_k)$ otherwise.

(Extended) Monopole-dimer model [2]

In the case of oriented Cartesian product of plane graphs each with a Pfaffian orientation, the loop-vertex model is known as the *monopole-dimer model* and the weight of a loop $\ell = (v_0, v_1, \dots, v_{2k-1}, v_{2k} = v_0)$ can be written *independent of the orientation* [2, Theorem 3.8]. In particular, for a plane graph, it can be expressed [1] as

$$w(\ell) = (-1)^{\text{number of vertices enclosed by } \ell} \prod_{j=0}^{2k-1} a_{v_j, v_{j+1}}.$$

Consequetly, $\det \mathcal{K}_G$ *becomes independent of the* k *Pfaffian orientations.*

High-dimensional cylindrical grid

An *ℓ-cylindrical grid* denoted $Q^{\ell}_{n_1,\dots,n_d}$ is the graph $C_{n_1} \square \cdots \square C_{n_\ell} \square P_{n_{\ell+1}} \square \cdots \square P_{n_d}$. For $\ell = 1$ ($\ell = d$), we call it a *cylindrical (toroidal) grid* and use the notation $Q^{\mathrm{Cyl}}_{n_1,\dots,n_d}$ ($Q^{\mathrm{Tor}}_{n_1,\dots,n_d}$).

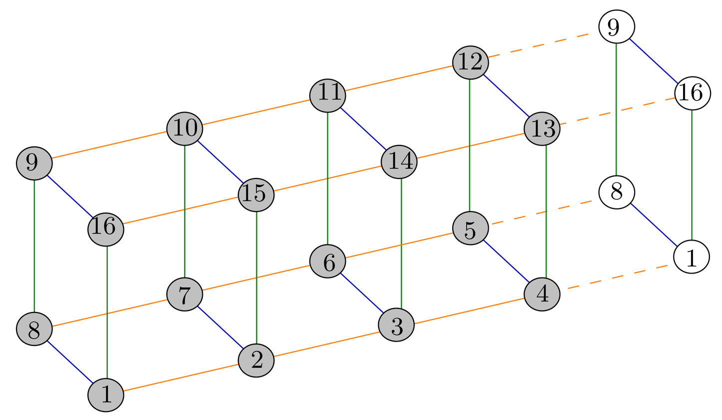

Figure: The boustrophedon labelling on the cylindrical grid $Q^{\mathrm{Cyl}}_{4,2,2}$.

Monopole-dimer model on cylindrical grid

Let G be the ℓ-cylindrical grid graph $Q^{\ell}_{2m_1,\dots,2m_d}$ with boustrophedon labelling in d dimension. Let $(G, \mathcal{O})$ be obtained from G by orienting the edges from a lower-labelled vertex to a higher-labelled vertex. Let the vertex weights be x for all vertices of G, and edge weights be $a_1, \dots, a_d$ for the edges along the different coordinate axes. Then the partition function of the monopole-dimer model on G is given by

$$\mathcal{Z}^{\mathrm{Mix}}_{2m_1,\dots,2m_d} = \prod_{i_1=1}^{m_1} \cdots \prod_{i_d=1}^{m_d} \left(x^2 + \sum_{s=1}^{\ell} 4a_s^2 \sin^2 \tfrac{(2i_s - 1)\pi}{2m_s} + \sum_{t=\ell+1}^{d} 4a_t^2 \cos^2 \tfrac{i_t \pi}{2m_t + 1} \right)^{2^{d-1}}.$$

High-dimensional Möbius grid

Let $Q_{n_1,\dots,n_d}$ be the d-dimensional grid graph, add an edge between the vertices $(1, k_2, \dots, k_d)$ and $(n_1, n_2 - k_2 + 1, \dots, n_d - k_d + 1)$ for all $1 \le k_i \le n_i$ $(2 \le i \le d)$ to obtain the d-dimensional *Möbius grid graph* and denoted as $Q^{\text{Möb}}_{n_1,\dots,n_d}$. We call these edges as *dashed edges* and the remaining as *solid edges*. Orient the solid edges from lower-labelled vertex to higher-labelled vertex, orient the dashed edge at 1 outward and the remaining dashed edges such that each two-dimensional square satisfies the clockwise-odd property.

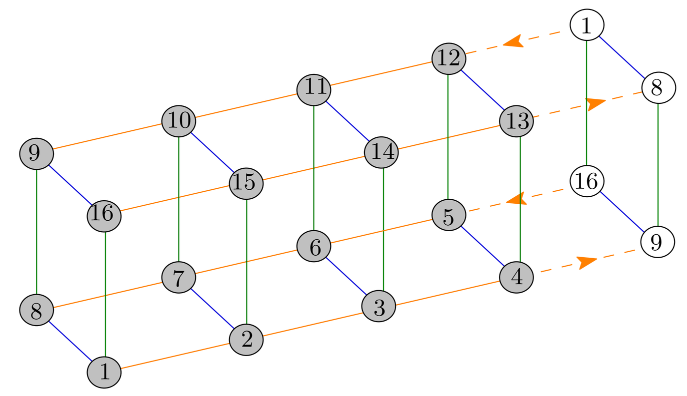

Figure: The three-dimensional Möbius grid graph $Q^{\text{Möb}}_{4,2,2}$.

Monopole-dimer model on Möbius grid

We define the *monopole-dimer model* on the d-dimensional Möbius grid graph G as the loop-vertex model on G with the above orientation $\mathcal{O}$. *The partition function* of the monopole-dimer model is then the partition function of the loop-vertex model.

Theorem

Let G be the three-dimensional Möbius grid graph $Q^{\text{Möb}}_{2m_1,2m_2,2m_3}$ with boustrophedon labelling. Let the vertex weights be x for all vertices of G, and edge weights be a_1, a_2 and a_3 for the edges along the x-, y- and z- coordinate axes respectively. Then the partition function of the monopole-dimer model on G is given by

$$\mathcal{Z}^{\text{Möb}}_{2m_1,2m_2,2m_3} = \prod_{i_1=1}^{m_1} \prod_{i_2=1}^{m_2} \prod_{i_3=1}^{m_3} \left(x^2 + 4a_1^2 \sin^2 \frac{(4i_1 - 1)\pi}{4m_1} + 4a_2^2 \cos^2 \frac{i_2 \pi}{2m_2 + 1} + 4a_3^2 \cos^2 \frac{i_3 \pi}{2m_3 + 1} \right)^4.$$

Relation between cylindrical and Möbius grid

Let $\mathcal{Z}^{\mathrm{Cyl}}_{4n_1,2n_2,2n_3}$ and $\mathcal{Z}^{\text{Möb}}_{2n_1,2n_2,2n_3}$ be the partition function of the monopole-dimer model on the three-dimensional Möbius grid $Q^{\text{Möb}}_{4n_1,2n_2,2n_3}$ and cylindrical grid $Q^{\mathrm{Cyl}}_{2n_1,2n_2,2n_3}$ with boustrophedon labelling, respectively. Then

$$\mathcal{Z}^{\mathrm{Cyl}}_{4n_1,2n_2,2n_3} = \left(\mathcal{Z}^{\text{Möb}}_{2n_1,2n_2,2n_3} \right)^2.$$

The three-dimensional Klein grid graph can be defined along the similar lines and the partition function of the monopole-dimer model on the Klein grid is given by

$$\mathcal{Z}^{\mathrm{Klein}}_{2m_1,2m_2,2m_3} = \prod_{i_1=1}^{m_1} \prod_{i_2=1}^{m_2} \prod_{i_3=1}^{m_3} \left(x^2 + 4a_1^2 \sin^2 \frac{(4i_1 - 1)\pi}{4m_1} + 4a_2^2 \sin^2 \frac{(2i_2 - 1)\pi}{2m_2} + 4a_3^2 \sin^2 \frac{(2i_3 - 1)\pi}{2m_3} \right)^4.$$

The product formula for Möbius and Klein does not generalise to higher dimensions.

References

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