Expansion Formulae for SL_3 Fock-Goncharov Cluster Algebras

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Cluster Algebra $\mathcal{A}_{\mathrm{SL}_3,m}$

Let P be a polygon and T be a triangulation on P, the initial quiver $Q_3(T)$ is defined as follows.

- Place two vertices on each arc of the triangulation T and one vertex in the interior of each triangle of T. The corresponding cluster variables are called **edge variables** and **face variables** respectively.
- Attach arrows to the vertices with clockwise oriented triangles.

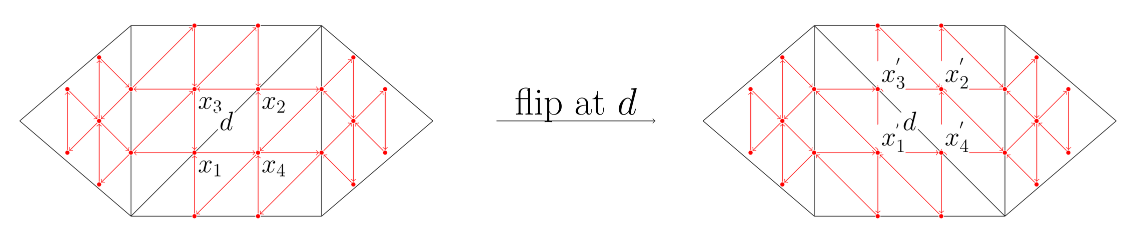

Figure 1: A hexagon P with triangulation T and the corresponding quiver $Q_3(T)$ (3-triangulation). A flip at the diagonal d is obtained by first mutating x_1 and x_2, and then mutating x_3 and x_4.

Plabic Graphs

Let $\hat{T}$ be the 3-triangulation associated to T. The plabic graph Γ is defined as follows.

- Place a white vertex on each vertex of $\hat{T}$.
- Place a black vertex in each of the three small triangles except for the internal one.
- Add black-to-white edges in each of the non-internal small triangles, making every black vertex trivalent.
- Remove all the vertices of degree one, and the resulting weighted graph is called the **weighted plabic graph associated to** $Q_3(T)$.

The defined plabic graph Γ is dual to the quiver $Q_3(T)$, in a way that every face of Γ corresponds to a vertex of $Q_3(T)$. For each quadrilateral face or hexagonal face F of Γ, define the **label** label(F) of the face to be the cluster variable sitting inside.

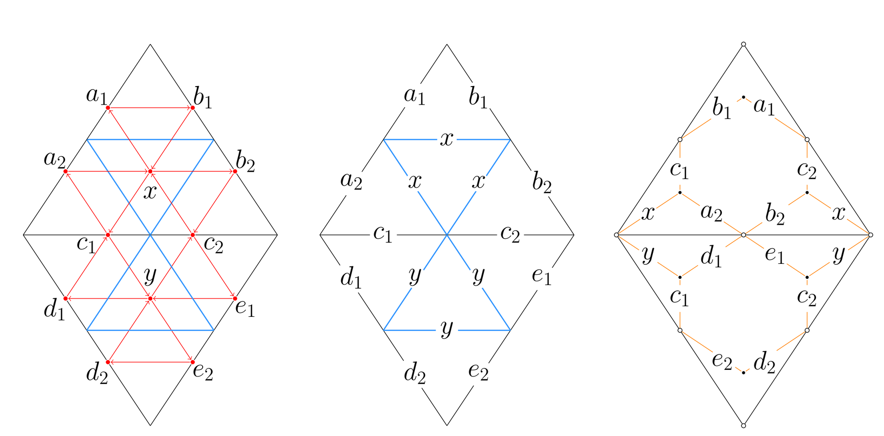

Figure 2: From left to right: the quiver $Q_3(T)$, the weighted 3-triangulation $\hat{T}$, the weighted plabic graph Γ. The weight of edges in Γ are labeled. And the label of a face of Γ is the cluster variable corresponding to the quiver vertex sitting inside the face.

Main Theorem

We shall display the statement of our main theorem below and explain the detailed definitions later.

Theorem

Let x be a cluster variable corresponding to a face or an edge, and let Γ_x be the corresponding plabic subgraph. Then the expansion of x in terms of the initial cluster is:

$$x = \frac{1}{\mathrm{label}(\Gamma_x)} \sum_{M \in \mathrm{D}(\Gamma_x)} \mathrm{wt}(M)\,\mathrm{ht}(M). \tag{1}$$

Plabic Subgraphs and Labels

- For each arc (i,j) on the polygon P, denote x_{ij} and x_{ji} the two **edge variables** corresponding to (i,j), where x_{ij} is the one that is placed closer to the vertex i.
- For each triangle (i,j,k) on P, denote x_{ijk} its corresponding **face variable**.

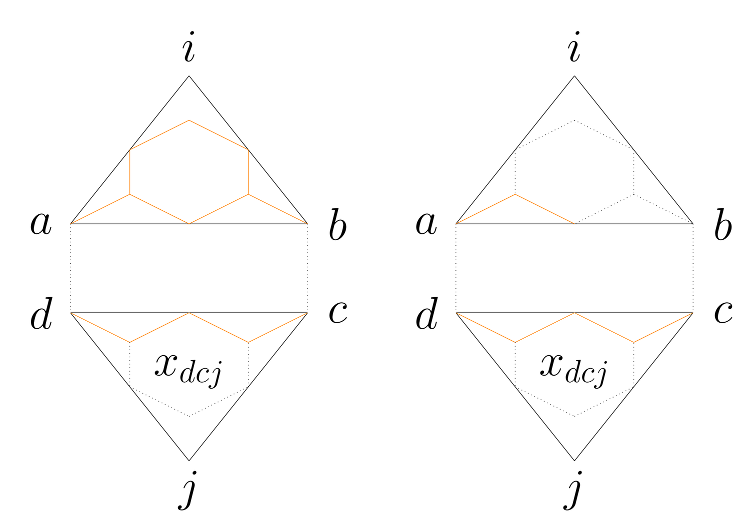

Figure 3: Left: The plabic subgraph $\Gamma_{x_{ij}}$. Right: The plabic subgraph $\Gamma_{x_{iaj}}$. Here only the first and last triangle are shown, and dashed edges being removed to form the plabic subgraph.

Consider a triangulation as described in Figure 3. The associated **plabic subgraphs** Γ_x and **labels** label(Γ_x) to the variables are defined as follows respectively.

- For an edge variable x_{ij}, define the corresponding Γ_{ij} to be the subgraph of Γ with a hexagonal face near vertex j removed. Define label$(\Gamma_{x_{ij}})$ by
$$\mathrm{label}(\Gamma_{x_{ij}}) := x_{dcj} \prod_{F \in \text{faces of } \Gamma_{x_{ij}}} x_F. \tag{2}$$
- For a face variable x_{iaj} such that (i,a) is an edge in T, we define Γ_{ija} to be the subgraph of Γ with 3 less faces: two hexagonal faces inside (i,a,b) and (j,d,c), and one quadrilateral face near vertex b. Define label$(\Gamma_{x_{ija}})$ by
$$\mathrm{label}(\Gamma_{x_{ija}}) := x_{dcj} \prod_{F \in \text{faces of } \Gamma_{x_{iaj}}} x_F. \tag{3}$$

Dimer Covers and Weights and Heights

A **dimer cover** M of a bipartite graph G is a collection of edges such that every vertex in G is incident to exactly on edge in M. We denote $\mathrm{D}(G)$ the set of all dimer covers of G.

There exists a unique dimer cover $M_0 \in \mathrm{D}(G)$ such that every boundary edge of M_0 is oriented counterclockwise from white to black, and exactly half the boundary edges of G are included in M_0. We call M_0 the **minimal dimer cover** .

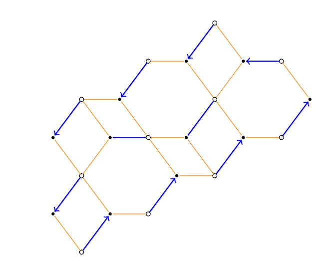

Figure 4: Example of a minimal dimer cover. The white-to-black counterclockwise orientation of boundary edges are shown.

The **weight** of a dimer cover M is defined to be the product of all edge weights in M.

Let $\overline{M} = M \cup M_0$, which pictorially is obtained by superimposing M on top of M_0. Then the **height** of M is

$$\mathrm{ht}(M) = \prod_{f \in \text{ cycles of } \overline{M}} \mathbf{y}_f \tag{4}$$

where the product is over the faces of G that are surrounded by a cycle of $\overline{M}$, and $\mathbf{y}_f$ is the coefficient corresponding to the cluster variable sitting in f.

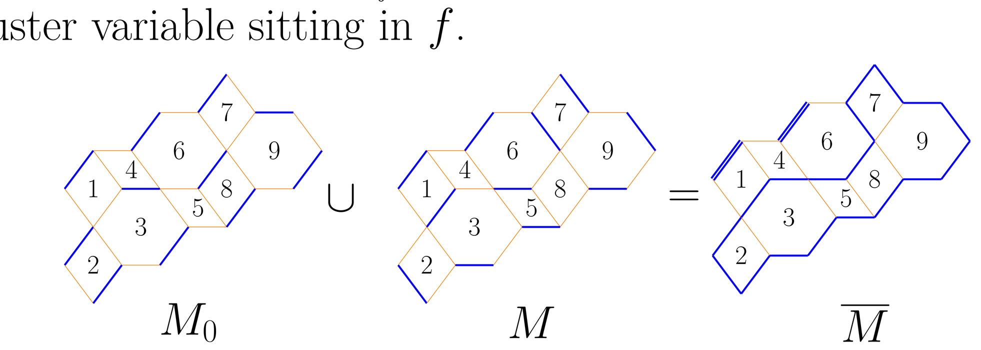

Figure 5: Superimposing a dimer cover M and the minimal dimer cover M_0. Face labels indicate the corresponding cluster variables and their coefficient. The height is $\mathrm{ht}(M) = \mathbf{y}_2\mathbf{y}_3\mathbf{y}_5\mathbf{y}_7\mathbf{y}_8\mathbf{y}_9$.

Poset structures

The F-polynomial corresponding to a cluster variable is defined by specializing all the x_i's to be 1. In terms of plabic graphs, the F-polynomial is the sum of heights of all dimer covers.

For G a plabic subgraph, define a poset $\mathcal{P}_{\mathrm{D}(G)}$ on $\mathrm{D}(G)$ as follows. For $M_1, M_2 \in \mathrm{D}(G)$, we have $M_1 < M_2$ if $\mathrm{ht}(M_1)$ is divisible by $\mathrm{ht}(M_2)$. This poset can also be constructed inductively. Fix M_0 as the minimal element, then every covering relation of the poset is given by **toggling** on faces.

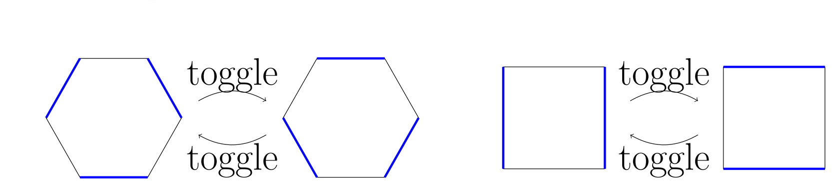

Figure 6: Illustration of toggle on a hexagon face and on a square face.

Theorem

Let $M \in \mathrm{D}(G)$, its height can be computed via $\mathcal{P}_{\mathrm{D}(G)}$ as follows. Take any chain from M_0 to M which corresponds to a sequence of toggles, then $\mathrm{ht}(M)$ is the product of the y-coefficients of the faces being toggled.

Note that the result does not depend on the specific choice of chains.

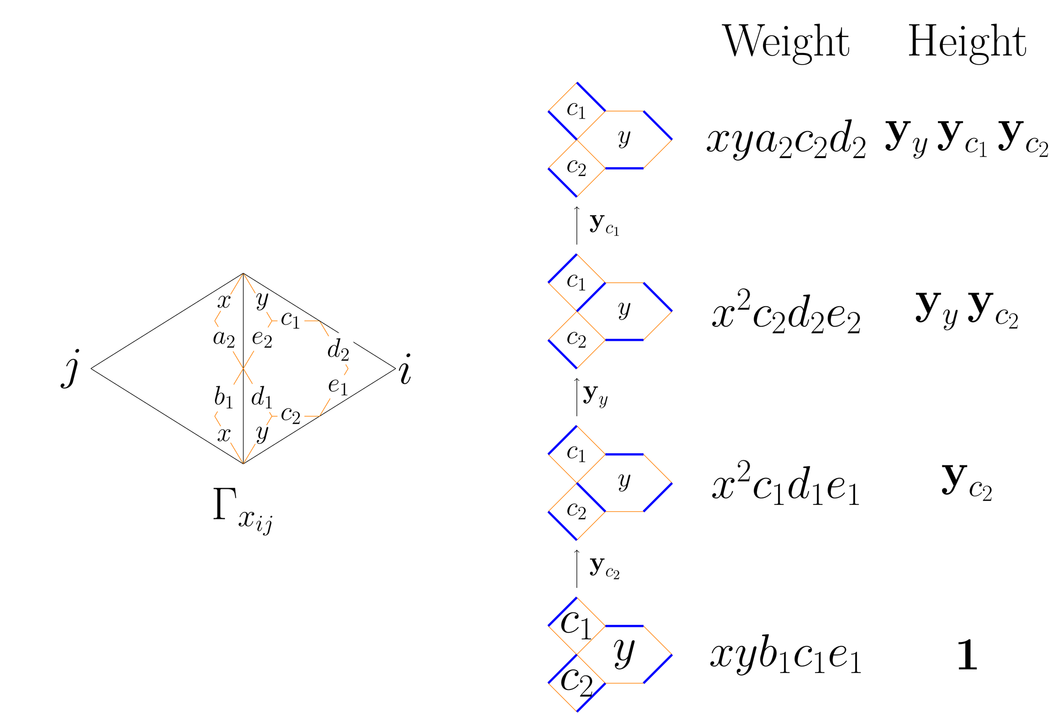

Figure 7:Left: the plabic subgraph $\Gamma_{x_{ij}}$. Right: the poset of all dimer covers on $\Gamma_{x_{ij}}$ and their corresponding weight and height. In this example, the expansion formula for the edge variable x_{ij} is

$$x_{ij} = \frac{1}{xyc_1c_2}(xb_1yc_1e + x^2d_1c_1e_1\mathbf{y}_{c_1} + x^2e_2d_2c_2\mathbf{y}_y\mathbf{y}_{c_2} + a_2xyd_2c_2\mathbf{y}_y\mathbf{y}_{c_1}\mathbf{y}_{c_2}).$$

The poset $\mathcal{P}_{\mathrm{D}(G)}$ is a distributive lattice via an argument of Propp [3], and its subposet consisting of join-irreducibles is isomorphic to a specific part of the quiver.

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