Turbulence Polyhedra

Relaxing Directedness and Acyclicity of Flows on Directed Acyclic Graphs

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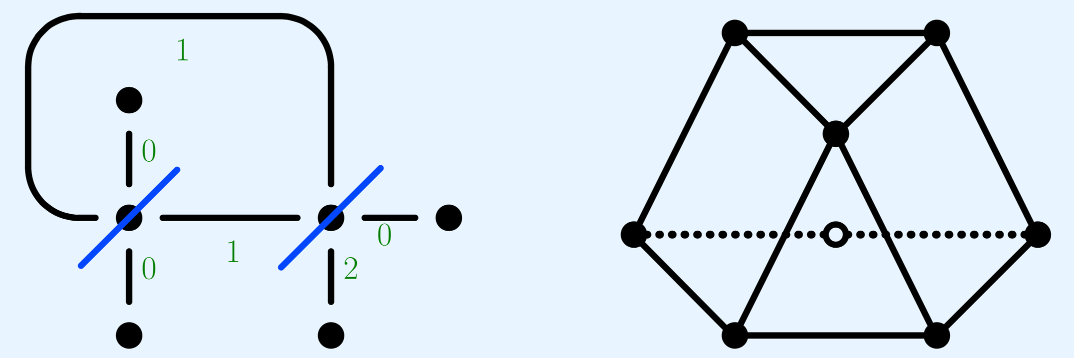
Definition: Turbulence Charts

An edge of a graph is **fringe** if it is the only edge incident to a vertex; otherwise, it is **internal**. We define a **turbulence chart** to be an undirected graph such that the half-edges incident each internal vertex are separated into two equivalence classes (by a blue line).



A unit flow on a turbulence chart is a labelling of the edges satisfying

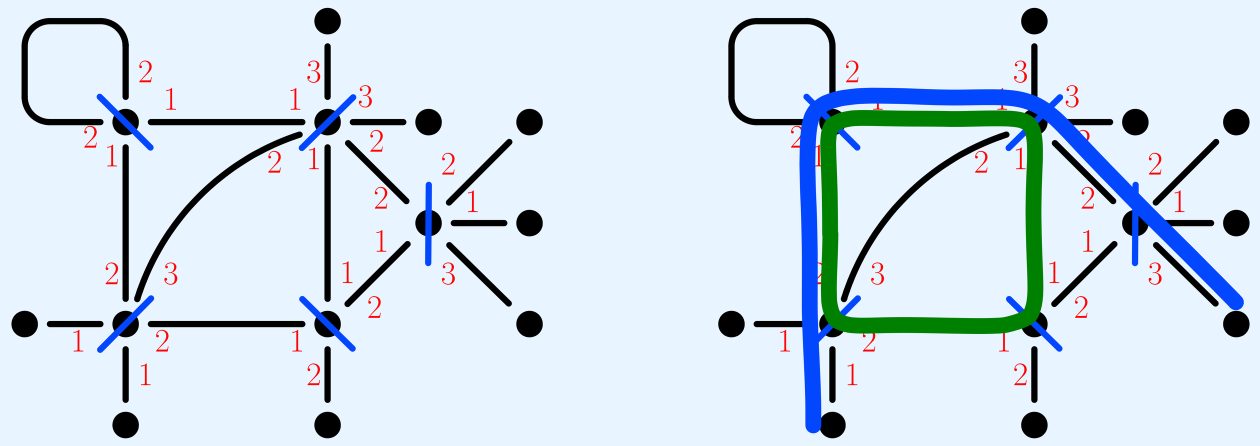
1. **(conservation of flow)** at every internal vertex, the sum of each equivalence class is the same
2. **(unit flow)** the sum of all fringe edges is 2.

A flow is labelled in green on the left. The **turbulence polyhedron** $\mathcal{F}_1(G)$ of a turbulence chart G is the polyhedron of unit flows, shown on the right.

We will obtain triangulation and subdivision results on turbulence polyhedra (middle column). These generalize existing triangulation results for flow polytopes and g-vector combinatorics of gentle algebras (right column).

Definition: Framings, Trails, Compatibility, Bundles

A framing on a turbulence chart is a total order on the half-edges of both equivalence classes at each internal vertex (drawn using red labels).



1. A **route** (blue) is a walk between two fringe vertices of the turbulence chart which crosses the equivalence class at each internal vertex (i.e, crosses the blue line). Routes are considered up to reversing the direction of the walk.
2. A **band** (green) is a cyclic walk crossing the equivalence class at each internal vertex, considered up to direction and cyclic equivalence.

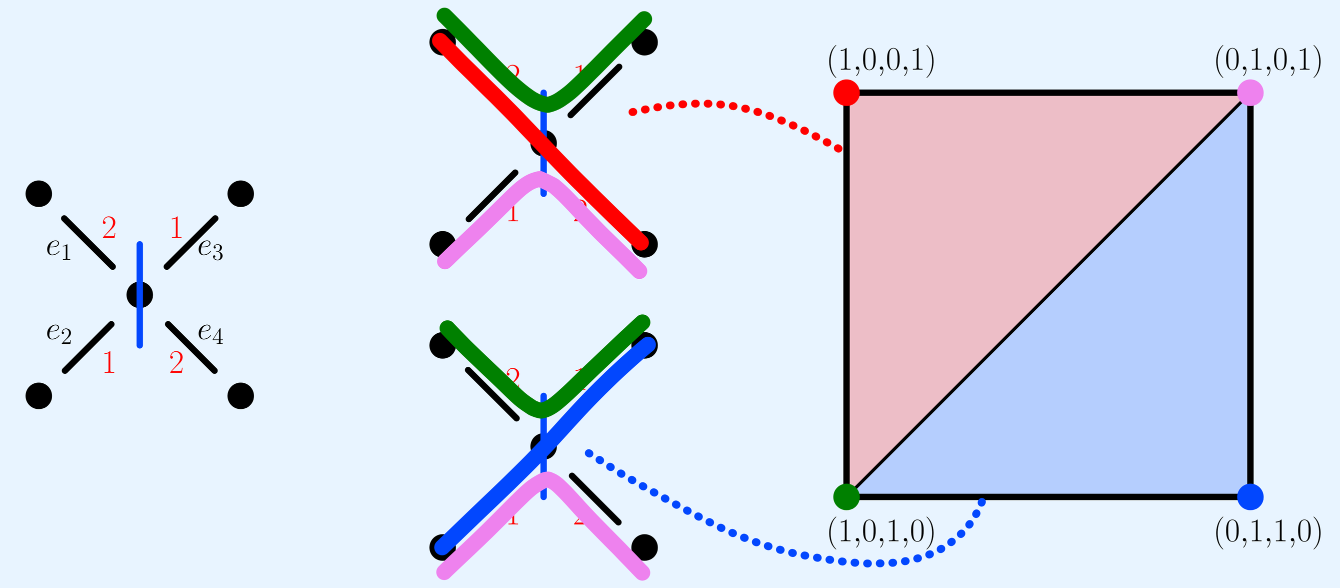
A **trail** is a route or band. Trails p and q are **incompatible** if they share a vertex or subwalk and WLOG p is above q with respect to the framing on both sides.

Trails are **compatible** if they are not incompatible. A **bundle** is a set of pairwise compatible trails. We will especially care about maximal bundles.

The **indicator vector** of a route is a unit flow. This allows us to consider a bundle of routes as a simplex in the turbulence polyhedron, giving us triangulations in the bounded case.

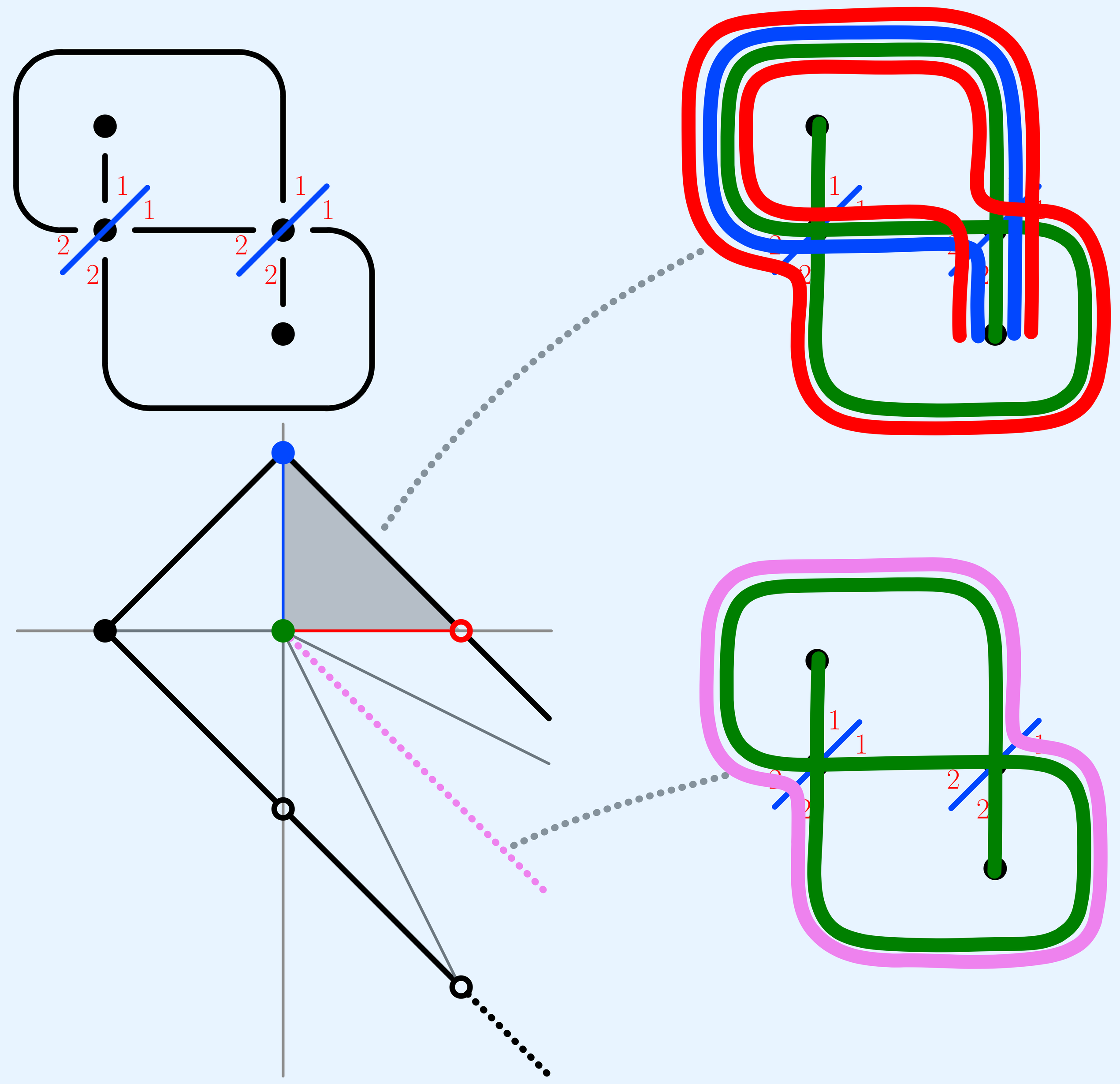
Theorem: Triangulations in the Bounded Case

Take a framed turbulence chart G with no bands. Then $\mathcal{F}_1(G)$ is a bounded polytope. Any maximal bundle C gives rise to a full-dimensional **bundle simplex** of $\mathcal{F}_1(G)$ whose vertices are the indicator vectors of routes of C. The collection of bundle simplices is a unimodular triangulation of $\mathcal{F}_1(G)$.



Theorem: Subdivisions in the Unbounded Case

Take a framed turbulence chart G containing at least one band. Then $\mathcal{F}_1(G)$ is an unbounded polyhedron. Any maximal bundle C gives rise to a **bundle space** of $\mathcal{F}_1(G)$ given by the Minkowski sum of the indicator vectors of routes of C with the indicator vectors of bands of C. The collection of bundle spaces is a subdivision covering all rational points of $\mathcal{F}_1(G)$.



Turbulence charts act as the common generalization of the combinatorics of **framed DAGs** and **gentle algebras**, which we discuss in this column.

Special Case: Framed DAGs

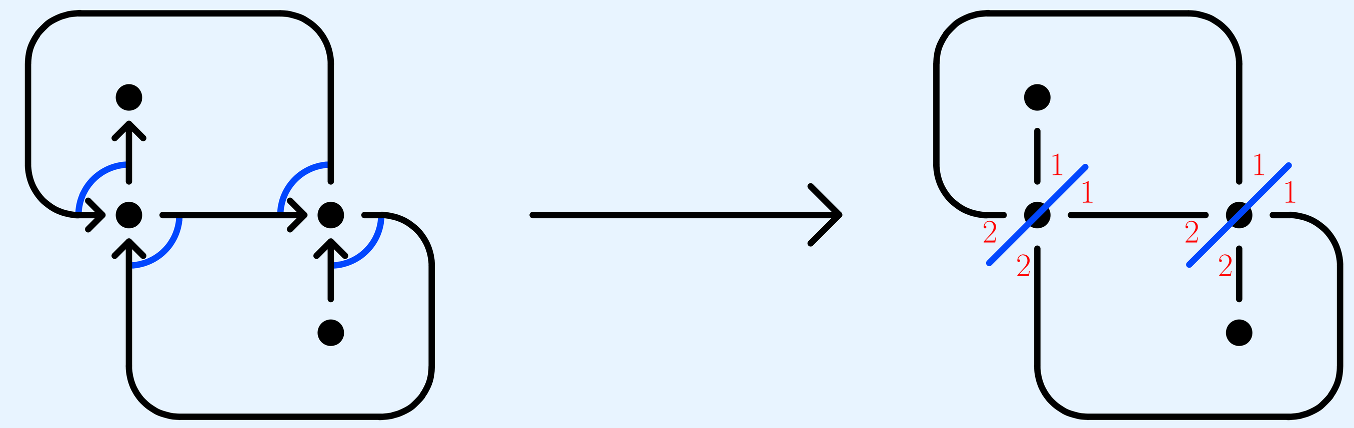
A **framed DAG** is a (D)irected (A)cyclic (G)raph with total orders in the incoming and outgoing edges of each internal vertex. Given a framed DAG (left), we can obtain a framed turbulence chart (right) by

1. reversing all outgoing framing orders,
2. separating the incoming from outgoing edges at each internal vertex, and
3. forgetting the direction of all edges.

Then the triangulated turbulence polyhedron is the same as the framing triangulation of the original flow polytope as introduced in [3] (see the unbounded example in the middle column).

Special Case: Gentle Algebras

Gentle algebras are an important class in the representation theory of algebras. The fringed quiver of a gentle algebra Λ (left) gives rise to a framed turbulence chart $C(\Lambda)$ by separating the arrow-heads from arrow-tails at each internal vertex and labelling arrow-heads with "2" and arrow-tails with "1."



The complex of compatible collections of routes on the turbulence chart $C(\Lambda)$ is the same as the non-kissing complex of Λ [2,4]. Accordingly, the turbulence polyhedron quotients onto the g-polyhedron of Λ, whose cone gives the g-vector fan (see the lower example in the middle column).

References

[1] Jonah Berggren. Flows on gentle algebras. *arXiv preprint*, 2025.

[2] Thomas Brüstle, Guillaume Douville, Kaveh Mousavand, Hugh Thomas, and Emine Yıldırım. On the combinatorics of gentle algebras. *Canad. J. Math*, 72(6):1551-1580, 2020.

[3] Vladimir I. Danilov, Alexander V. Karzanov, and Gleb A. Koshevoy. Coherent fans in the space of flows in framed graphs. In *24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012)*, volume AR of *Discrete Math. Theor. Comput. Sci. Proc.*, pages 481–490. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2012.

[4] Yann Palu, Vincent Pilaud, and Pierre-Guy Plamondon. Non-kissing complexes and tau-tilting for gentle algebras. *Mem. Amer. Math. Soc.*, 274(1343):vii+110, 2021.

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