Characteristic polynomials of deformations of Coxeter arrangements via levels of regions

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Deformations of Coxeter arrangements

The study of Coxeter arrangements plays an essential role in the theory of hyperplane arrangements. Coxeter arrangements arise from the reflecting hyperplanes associated with the root systems of finite Coxeter groups, which reveal the symmetries and combinatorial structures in a geometric context.

Deformations of Coxeter arrangements are affine arrangements where each hyperplane is parallel to some hyperplane of the original arrangement. Numerous types of deformations have been extensively studied over the years, including the Catalan arrangements and the Shi arrangements, particularly concerning characteristic polynomials and the enumeration of regions.

The Coxeter arrangement of type A_{n-1} in $\mathbb{R}^n$ is $\mathrm{Cox}_{\mathcal{A}}(n) = \{x_i - x_j = 0 \mid 1 \le i \ne j \le n\}$.

We define the **non-degenerate deformation** of the Coxeter arrangement of type A_{n-1}

$$\mathcal{A} = \{x_i - x_j = a_{ij}^{(1)}, \dots, a_{ij}^{(t_{ij})} \mid 1 \le i \ne j \le n\}, \tag{1}$$

where $a_{ij}^{(1)}, \dots, a_{ij}^{(t_{ij})} \in \mathbb{R}$ and $t_{ij} \ge 1$ for all $1 \le i, j \le n$.

The Coxeter arrangement of type B_n in $\mathbb{R}^n$ is $\mathrm{Cox}_{\mathcal{B}}(n) = \{x_i = 0 \mid 1 \le i \le n\} \cup \{x_i \pm x_j = 0 \mid 1 \le i, j \le n\}$.

Similarly, we define the **non-degenerate deformation** of the Coxeter arrangement of type B_n

$$\mathcal{B} = \{x_i = a_i^{(1)}, \dots, a_i^{(r_i)} \mid 1 \le i \le n\} \cup \{x_i - x_j = b_{ij}^{(1)}, \dots, b_{ij}^{(s_{ij})} \mid 1 \le i \ne j \le n\} \cup \{x_i + x_j = c_{ij}^{(1)}, \dots, c_{ij}^{(t_{ij})} \mid 1 \le i \ne j \le n\}, \tag{2}$$

where $a_i^{(1)}, \dots, a_i^{(r_i)}, b_{ij}^{(1)}, \dots, b_{ij}^{(s_{ij})}, c_{ij}^{(1)}, \dots, c_{ij}^{(t_{ij})} \in \mathbb{R}$ and $r_i, s_{ij}, t_{ij} \ge 1$ for all $1 \le i, j \le n$.

In this paper, we establish a formula for the characteristic polynomial of general deformations of Coxeter arrangements, expanding the polynomial into terms related to the numbers of regions with different levels.

Characteristic polynomials

A **hyperplane arrangement** $\mathcal{A} = \{H_1, \dots, H_m\}$ is a finite set of affine hyperplanes in $\mathbb{R}^n$. The **intersection poset** $L(\mathcal{A})$ of arrangement $\mathcal{A}$ is the set of all nonempty intersections of hyperplanes in $\mathcal{A}$, including $\mathbb{R}^n$ itself, partially ordered by reverse inclusion.

The **characteristic polynomial** $\chi_{\mathcal{A}}(t)$ of a hyperplane arrangement $\mathcal{A}$ is defined by

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) t^{\dim(x)},$$

where $L(\mathcal{A})$ is the intersecting poset of $\mathcal{A}$, and $\mu(x) = \mu(\hat{0}, x)$ is the Möbius function of $L(\mathcal{A})$.

Given a hyperplane $H_0 \in \mathcal{A}$, define the **restriction arrangement** $\mathcal{A}^{H_0}$ in the affine subspace $H_0 \cong \mathbb{R}^{n-1}$ by

$$\mathcal{A}^{H_0} = \{H_0 \cap H \ne \emptyset : \ H \in \mathcal{A} - \{H_0\}\}.$$

Let $\mathcal{A}' = \mathcal{A} - \{H_0\}$ and $\mathcal{A}'' = \mathcal{A}^{H_0}$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a **triple of arrangements with distinguished hyperplane** H_0. The well-known Deletion-Restriction Lemma [4] shows a recursive property for the characteristic polynomials of a triple of arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ as follows:

$$\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t).$$

Levels of regions

A **region** of an arrangement $\mathcal{A}$ is a connected component of the complement of the hyperplanes. Let $\mathcal{R}(\mathcal{A})$ denote the set of regions of $\mathcal{A}$, and let $r(\mathcal{A}) = |\mathcal{R}(\mathcal{A})|$ denote the number of regions in the arrangement $\mathcal{A}$.

The following definition may not be as familiar to the audience. Given a subset $X \subset \mathbb{R}^n$, the **level** of X is the smallest non-negative integer ℓ such that

$$X \subset B(W, r) = \{x \in \mathbb{R}^n : d(x, W) \le r\},$$

for some subspace W of dimension ℓ and a real number $r > 0$. Informally speaking, the level of a region equals the dimension of its unbounded directions, reflecting the region's degree of freedom. Let $\mathcal{R}_\ell(\mathcal{A})$ denote the collection of regions of $\mathcal{A}$ with level ℓ, and let $r_\ell(\mathcal{A}) = |\mathcal{R}_\ell(\mathcal{A})|$.

For example, let $\mathcal{A} = \{H_1, H_2, H_3, H_4\}$ be a hyperplane arrangement in $\mathbb{R}^2$, see Figure 1, where

$$H_1 : x = 0, \ H_2 : y = 0, \ H_3 : x + y = 1, \ H_4 : y = 1.$$

For the three regions labeled, we show that $\ell(\Delta_0) = 0$, $\ell(\Delta_1) = 1$, $\ell(\Delta_2) = 2$.

Moreover, we have the number of regions with each level

$$r_0(\mathcal{A}) = 1, r_1(\mathcal{A}) = 2, r_2(\mathcal{A}) = 6,$$

and $r(\mathcal{A}) = r_0(\mathcal{A}) + r_1(\mathcal{A}) + r_2(\mathcal{A}) = 9$ in arrangement $\mathcal{A}$.

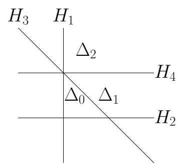

Figure 1:The hyperplane arrangement $\mathcal{A}$

Main results

The following is our main result, a new expansion of the characteristic polynomials of deformations of Coxeter arrangements.

Theorem(type A)

Let $\mathcal{A}$ be a non-degenerate deformation of $\mathrm{Cox}_A(n)$ as in Equation (1). Then,

$$\chi_{\mathcal{A}}(t) = \sum_{k=0}^{n} (-1)^{n-k} \cdot r_k(\mathcal{A}) \cdot \binom{t}{k},$$

where $r_k(\mathcal{A})$ is the number of regions with level k in arrangement $\mathcal{A}$.

The above result can be extended to type B deformations as well.

Theorem(type B)

Let $\mathcal{B}$ be a non-degenerate deformation of $\mathrm{Cox}_B(n)$ as in Equation (2). Then,

$$\chi_{\mathcal{B}}(t) = \sum_{k=0}^{n} (-1)^{n-k} \cdot r_k(\mathcal{B}) \cdot \binom{\frac{t-1}{2}}{k},$$

where $r_k(\mathcal{B})$ is the number of regions with level k in arrangement $\mathcal{B}$.

Example 1. Let $\mathcal{A} = \{H_1, H_2, H_3, H_4, H_5\}$ be a hyperplane arrangement in $\mathbb{R}^3$, where

$$H_1 : x_1 - x_2 = 0, \ H_2 : x_1 - x_2 = 1, \ H_3 : x_2 - x_3 = 0, \ H_4 : x_1 - x_3 = 1, \ H_5 : x_1 - x_3 = 0.$$

Figure 2 shows the projection of the arrangement $\mathcal{A}$ onto the plane $x_1 + x_2 + x_3 = 0$, where all the regions are labeled by their levels. The characteristic polynomial

$$\chi_{\mathcal{A}}(t) = t^3 - 5t^2 + 6t = 6\binom{t}{3} - 4\binom{t}{2} + 2\binom{t}{1},$$

where $r_3(\mathcal{A}) = 6$, $r_2(\mathcal{A}) = 4$, and $r_1(\mathcal{A}) = 2$.

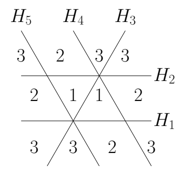

Figure 2:An example of type A.

Example 2. Let $\mathcal{B} = \{H_1, H_2, H_3, H_4\}$ be a hyperplane arrangement in $\mathbb{R}^2$ shown in Figure 3, where

$$H_1 : x_1 = 0, \ H_2 : x_1 - x_2 = 0, \ H_3 : x_2 = 0, \ H_4 : x_1 + x_2 = 1.$$

The characteristic polynomial

$$\chi_{\mathcal{B}}(t) = t^2 - 4t + 5 = 8\binom{\frac{t-1}{2}}{2} + 2\binom{\frac{t-1}{2}}{0},$$

where $r_2(\mathcal{B}) = 8$, $r_1(\mathcal{B}) = 0$, and $r_0(\mathcal{B}) = 2$.

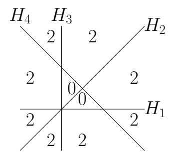

Figure 3:An example of type B.

The result of type A generalizes several recent results by Chen et al. (see Theorem 1.5 of [3] and Theorem 1.2 of [2]) on the characteristic polynomials of a specific type of arrangements, including Catalan-type arrangements and semiorder-type arrangements. While their proof relies on certain symmetries of coefficients, our results employs a more general method and requires much fewer restrictions on arrangements.

Moreover, our theorems provide a novel approach to determine the characteristic polynomial of hyperplane arrangement by counting the regions with fixed levels.

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