A Toric Analogue for Greene's Rational Function of a Poset

Elise Catania
University of Minnesota

Greene's Rational Function of a Poset

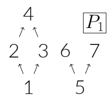
Motivated by a combinatorial proof of the Murnaghan–Nakayama formula, C. Greene associated to every poset P on $[n] = \{1, 2, \ldots, n\}$ a rational function

$$\Psi^P(\mathbf{x}) = \sum_{w \in \mathcal{L}(P)} \frac{1}{(x_{w_1} - x_{w_2})(x_{w_2} - x_{w_3}) \cdots (x_{w_{n-1}} - x_{w_n})}.$$

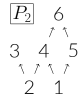
Here $\mathcal{L}(P)$ denotes the set of linear extensions $w = (w_1 < \cdots < w_n)$ of P.

Example. We evaluate Greene's rational function for two posets.

$\Psi^{P_1}(\mathbf{x}) = 0$



$\Psi^{P_2}(\mathbf{x}) = \frac{x_1 - x_6}{(x_2-x_3)(x_2-x_4)(x_1-x_4)(x_1-x_5)(x_4-x_6)(x_5-x_6)}$



Properties of $\Psi^P(\mathbf{x})$

Theorem (Greene). For a strongly planar poset, if $H(P)$ is disconnected, the function $\Psi^P(\mathbf{x})$ vanishes and otherwise, we have

$$\Psi^P(\mathbf{x}) = \frac{\prod_{\delta \in \Delta}(x_{\min(\delta)} - x_{\max(\delta)})}{\prod_{i \lessdot_P j}(x_i - x_j)},$$

where Δ is the set of bounded regions of $H(P)$.

Theorem (Boussicault). For any poset, $H(P)$ is disconnected iff $\Psi^P(\mathbf{x}) = 0$.

Theorem (Boussicault–Féray–Lascoux–Reiner). For a connected poset, the minimal denominator of $\Psi^P(\mathbf{x})$ is $\prod_{i \lessdot_P j}(x_i - x_j)$.

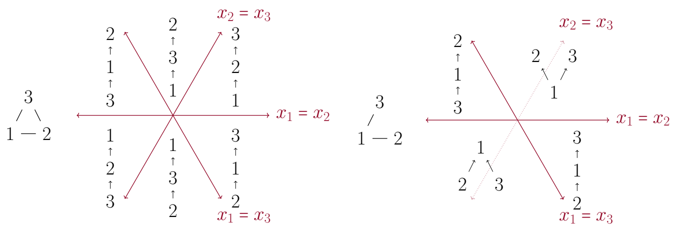
Posets and Graphic Hyperplane Arrangements

A poset P on $[n]$ gives rise to an open polyhedral cone $c(P)$ in $\mathbb{R}^n$, where

$$c(P) := \{x \in \mathbb{R}^n : x_i < x_j \text{ if } i <_P j\}.$$

Let G be a simple, undirected graph on the vertex set $[n]$. Then, the *graphic hyperplane arrangement* $\mathcal{A}(G)$ is defined to be $\mathcal{A}(G) := \bigcup_{\{i,j\} \in G} \mathcal{H}_{ij}$ where $\mathcal{H}_{ij}$ is the hyperplane $x_i = x_j$.

Example. Pictures drawn within the 2-plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$.



Chambers in $\mathbb{R}^n - \mathcal{A}(G)$ biject with acyclic orientations of G.

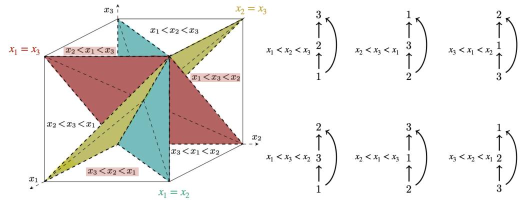
Toric Posets and Toric Graphic Hyperplane Arrangements

Let $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. The *toric graphic hyperplane arrangement* associated to G is

$$\mathcal{A}_{\text{tor}}(G) = \pi(\mathcal{A}(G)).$$

A connected component of $\mathbb{R}^n/\mathbb{Z}^n - \mathcal{A}_{\text{tor}}(G)$ is a *toric chamber*.

Example.



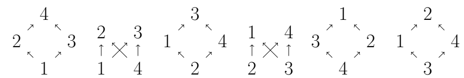
The uncolored version of the cube on the left is from [4].

Definition. If quivers Q_1, Q_2 differ by converting a source to a sink or vice versa, then they differ by a *flip*.

Theorem (Develin–Macauley–Reiner). Toric chambers of $\mathcal{A}_{\text{tor}}(G)$ biject with flip-equivalence classes of acyclic quivers having underlying graph G.

Definition. A *toric poset* $[Q]$ is a flip-equivalence class of acyclic quivers.

Example.



Definition. For a toric poset $[Q]$, the set of toric total extensions $\mathcal{L}_{\text{tor}}([Q])$ is defined as $\mathcal{L}_{\text{tor}}([Q]) := \{[w] : w \in \mathcal{L}(Q') \text{ for some } Q' \in [Q]\}$.

Toric Analogue

Definition. Let $[Q]$ be a toric poset. Then, we define $\Psi_{\text{tor}}^{[Q]}(\mathbf{x})$ as

$$\Psi_{\text{tor}}^{[Q]}(\mathbf{x}) := \sum_{[w] \in \mathcal{L}_{\text{tor}}([Q])} \Psi_{\text{tor}}^{[w]}(\mathbf{x}), \text{ where}$$

$$\Psi_{\text{tor}}^{[w]}(\mathbf{x}) = \frac{1}{(x_{w_1} - x_{w_2})(x_{w_2} - x_{w_3}) \cdots (x_{w_{n-1}} - x_{w_n})(x_{w_n} - x_{w_1})}.$$

Recovering the Kleiss-Kuijf Shuffle Relations

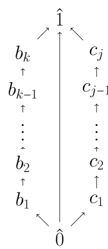
Theorem. Let P be a bounded, strongly planar poset with $\hat{0}, \hat{1}$. Let Q be the quiver resulting from adding the edge $\hat{0} \to \hat{1}$ in $H(P)$. Then,

$$\Psi_{\text{tor}}^{[Q]}(\mathbf{x}) = \frac{1}{x_{\hat{1}} - x_{\hat{0}}} \frac{\prod_{\delta \in \Delta}(x_{\min(\delta)} - x_{\max(\delta)})}{\prod_{i \lessdot_P j}(x_i - x_j)}.$$

Let $\mathbf{b} = (b_1, b_2, \ldots, b_k)$ and $\mathbf{c} = (c_1, c_2, \ldots, c_j)$.

Corollary (Kleiss-Kuijf Relations). For $\Psi_{\text{tor}}^{[Q]}(\mathbf{x})$ where $[Q]$ contains the quiver to the right,

$$\Psi_{\text{tor}}^{[Q]}(\mathbf{x}) = \sum_{\mathbf{a} \in \mathbf{b} \sqcup\!\sqcup \mathbf{c}} \Psi_{\text{tor}}^{[(\hat{1}, \hat{0}, \mathbf{a})]}(\mathbf{x}) = (-1)^k \Psi_{\text{tor}}^{[(\hat{1}, \text{rev}(\mathbf{b}), \hat{0}, \mathbf{c})]}(\mathbf{x}).$$



Properties of $\Psi_{\text{tor}}^{[Q]}(\mathbf{x})$

Theorem. Let $[Q]$ be a toric poset and G the underlying graph of $[Q]$. If G is disconnected with at least three vertices or has a cut vertex, then $\Psi_{\text{tor}}^{[Q]}(\mathbf{x}) = 0$.

Theorem. For $[Q]$ a toric poset, $\Psi_{\text{tor}}^{[Q]}(\mathbf{x})$ can be expressed over denominator

$$\prod_{\{i,j\} \in [Q]_{\text{Hasse}}} (x_i - x_j)$$

where we take the product over all edges $\{i, j\}$ in $[Q]_{\text{Hasse}}$.

A Recursive Algorithm for Finding Toric Total Extensions

Theorem. Let a, b be two torically incomparable elements in the toric poset $[Q]$.

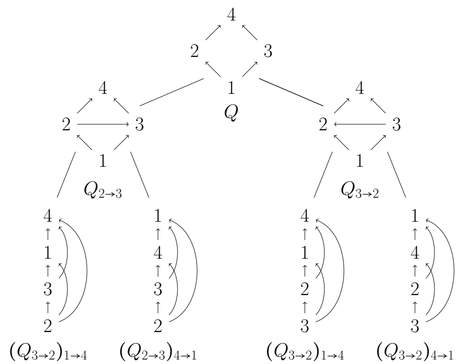
(i) For a, b in different components of the graph of $[Q]$, then $[Q_{a \to b}] = [Q_{b \to a}]$ and

$$\mathcal{L}_{\text{tor}}([Q]) = \mathcal{L}_{\text{tor}}([Q_{a \to b}]) = \mathcal{L}_{\text{tor}}([Q_{b \to a}]).$$

(ii) Assume a, b are distance two in the graph of the toric transitive closure $\overline{[Q]}$, say both adjacent to the vertex v. Then if one chooses $Q' \in \overline{[Q]}_v$, that is, Q' is a representative of $\overline{[Q]}$ with v a source, we have

$$\mathcal{L}_{\text{tor}}([Q]) = \mathcal{L}_{\text{tor}}([Q'_{a \to b}]) \sqcup \mathcal{L}_{\text{tor}}([Q'_{b \to a}]).$$

Example.



Reading the leaves left-to-right,

$$\mathcal{L}_{\text{tor}}([Q]) = \{[(1,4,2,3)], [(1,2,3,4)], [(1,4,3,2)], [(1,3,2,4)]\}.$$

The following theorem is a key component in the proof of our algorithm.

Theorem. When Q_1, Q_2 are flip-equivalent acyclic quivers having vertex v as a source, they are flip-equivalent by a flip sequence keeping v a source throughout.

References

[1] A. Boussicault. Action du groupe symétrique sur certaines fractions rationnelles suivi de puissances paires du vandermonde, 2009.
[2] A. Boussicault, V. Féray, A. Lascoux, and V. Reiner. Linear extension sums as valuations of cones. *Journal of Algebraic Combinatorics*, 35: 573–610, 2010.
[3] E. Catania. A toric analogue for greene's rational function of a poset, 2024. arXiv:2409.04907.
[4] M. Develin, M. Macauley, and V. Reiner. Toric partial orders. *Transactions of the American Mathematical Society*, 368(4):2263–2287, 2016.
[5] C. Greene. A rational-function identity related to the Murnaghan–Nakayama formula for the characters of S_n. *Journal of Algebraic Combinatorics*, 1(3):235–255, 1992.
[6] R. Kleiss and H. Kuijf. Multigluon cross sections and 5-jet production at hadron colliders. *Nuclear Physics B*, 312(3):616–644, 1989.

Contact Information: catan042@umn.edu