Gröbner bases and the Lefschetz properties for powers of a general linear form in the squarefree algebra

Joint with *Samuel Lundqvist, Fatemeh Mohammadi, Matthias Orth* and *Eduardo Sáenz-de-Cabezón*

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Goal of the poster

Consider the polynomial ring $R = \mathbf{k}[x_1, \ldots, x_n]$ over a field $\mathbf{k}$ of characteristic zero.

We want to determine the reduced Gröbner basis of all ideals of the form

$$I_{n,k} = (x_1^2, \ldots, x_n^2, (x_1 + \cdots + x_n)^k).$$

This is done by finding a candidate for each Gröbner basis and then showing that we have the right number of polynomials.

Some commutative algebra

Let $\prec$ be a monomial order on R. The *initial ideal* $\mathrm{in}(I)$ of a ideal I of R is the monomial ideal generated by the largest monomials w.r.t. $\prec$ of all $f \in I$. A *Gröbner basis* for I is a finite collection of polynomials $G = \{g_1, \ldots, g_m\}$ in I such that $(\mathrm{in}(G)) = \mathrm{in}(I)$.

The *Hilbert series* of an algebra $A = \mathbf{k}[x_1, \ldots, x_n]/I$ is the generating series for the dimensions of its graded pieces,

$$\mathrm{HS}(A;t) = \sum_{i=0}^{\infty} \dim_{\mathbf{k}}(A_i)t^i.$$

In all cases here, A will be artinian, i.e. $\dim_{\mathbf{k}}(A_i) = 0$ for $i \gg 0$, so $\mathrm{HS}(A;t)$ will be a polynomial. It is known that

$$\mathrm{HS}(R/I;t) = \mathrm{HS}(R/\mathrm{in}(I);t).$$

Since $\mathrm{in}(I)$ is a monomial ideal, calculating $\mathrm{HS}(R/\mathrm{in}(I);t)$ becomes a counting question.

An artinian algebra $A = \mathbf{k}[x_1, \ldots, x_n]/I$ has the *strong Lefschetz property* (SLP) if there is a homogeneous polynomial $\ell \in A_1$ of degree one such that all multiplication maps

$$\cdot\ell^k : A_i \to A_{i+k}$$

sending f to $\ell \cdot f$, are either injective or surjective for all $i, k \geq 0$. If it holds for $k = 1$, we say A has the *weak Lefschetz property* (WLP).

Example 1. By counting squarefree monomials, we have for $A = R/(x_1^2, \ldots, x_n^2)$ that $\mathrm{HS}(A;t) = (1+t)^n$. Further, A has the SLP iff for all $k \geq 2$ we have

$$\mathrm{HS}(A/(x_1 + \cdots + x_n)^k;t) = \mathrm{HS}(R/I_{n,k};t) = \left[(1-t^k)(1+t)^n\right]$$

where the brackets indicate truncation at the first non-positive coefficient.

Main result

The answer to our main question is the following description.

Theorem 2 ([JKLM+24]). *The reduced Gröbner basis of $I_{n,k}$ for $k \geq 2$ is given by*

$$G_{n,k} = \{x_1^2, \ldots, x_n^2\} \cup \bigcup_{d=k}^{k+\lfloor (n-k)/2 \rfloor} \{g_{A,n,k} \mid A \in \mathcal{A}, |A| = d\},$$

where $\mathcal{A}$ is the family of subsets $A \subseteq \{1, \ldots, n\}$ satisfying $\max(A) = 2|A| - k$, and minimal with respect to inclusion. For $|A| = d$,

$$g_{A,n,k} = e_d(x_{i_1}, \ldots, x_{i_{n-d+k}}) \tag{1}$$

is the elementary symmetric polynomial of degree d in the variables indexed by the set $\{i_1, \ldots, i_{n-d+k}\} = A \cup \{2d-k+1, \ldots, n\}$.

To show each $g_{A,n,k} \in I_{n,k}$, we use the following relation.

Proposition 3. *Let $f_{S,n,k} \in I_{n,k}$ for $S \subseteq [n]$ be the squarefree part of the polynomial $(\prod_{i \in S} x_i)(x_1 + \cdots + x_n)^k$. Then the elements $g_{A,n,k}$ can be written as*

$$g_{A,n,k} = \sum_{i=0}^{d-k} (-1)^i \frac{k}{(k+i)\binom{d}{k+i}} \sum_{S \in \mathcal{T}_i(A)} f_{S,n,k}$$

where $d = |A|$ and

$$\mathcal{T}_i(A) = \{S \subseteq [n] : |S| = d-k \text{ and } |S \cap (\{1, \ldots, 2d-k\} \setminus A)| = i\}.$$

The proof of the above relation relies on

$$\sum_{i=0}^{j} (-1)^i \binom{j}{i} \binom{k+i-1}{j-1} = 0$$

for all $j, k \geq 1$, a fun exercise for anyone interested.

Lattice paths

To conclude that $G_{n,k}$ is a Gröbner basis for $I_{n,k}$, it suffices to show that $\mathrm{HS}(R/(\mathrm{in}(G_{n,k}))) = \left[(1-t^k)(1+t)^n\right]$. If so, since it is known that the Hilbert series for $R/I_{n,k}$ cannot be smaller than this coefficient-wise, we have

$$\begin{aligned}\left[(1-t^k)(1+t)^n\right] &\leq \mathrm{HS}(R/I_{n,k};t) = \mathrm{HS}(R/\mathrm{in}(I_{n,k});t)\\ &\leq \mathrm{HS}(R/(\mathrm{in}(G_{n,k}))) = \left[(1-t^k)(1+t)^n\right]\end{aligned}$$

thereby proving that $(\mathrm{in}(G_{n,k})) = \mathrm{in}(I_{n,k})$ and that $R/(x_1^2, \ldots, x_n^2)$ has the SLP. This equality of Hilbert series is done via a lattice path bijection.

Definition 4. An (N,E)*-lattice path* is a path on the lattice $\mathbb{Z}^2$ that begins at $(0,0)$ and consists only of northward steps (in the direction $(0,1)$, denoted N) and eastward steps (in the direction $(1,0)$, denoted E). For example,

$$NEENEN \leftrightarrow x_1x_4x_6.$$

There exists a bijection that maps an (N,E)-lattice path of length n taking d steps north to the squarefree monomial $\prod_{j \in J} x_j$ of degree d where the subset $J \subseteq [n]$ contains an index j if and only if the j-th step in the path is north.

Proposition 5. *We have the following necessary result for the lattice paths.*

1. *A lattice path intersects the line $y = x + k$ if and only if the corresponding monomial is divisible by some $\mathrm{in}(g_{A,n,k})$ where $g_{A,n,k} \in G_{n,k}$.*
2. *Monomials outside $\mathrm{in}(G_{n,k})$ are in bijection with lattice paths never touching $y = x + k$ taking exactly n steps.*
3. *If $2d - k < n$, the number of lattice paths taking d steps north and touching $y = x + k$ is $\binom{n}{d-k}$.*

Here 1. is a translation of $\max(A) = 2|A| - k$ for $g_{A,n,k}$ to lattice paths and 3. can be proven via an induction on the length of the lattice path. Since there are $\binom{n}{d}$ lattice paths taking d steps north and n steps in total, 2. gives that the coefficient of t^d in $\mathrm{HS}(R/(\mathrm{in}(G_{n,k})))$ is $\binom{n}{d} - \binom{n}{d-k}$ when $2d - k < n$. Since $\left[(1-t^k)(1+t)^n\right]$ has the same coefficient of t^d for those d, this proves the main part of $\mathrm{HS}(R/(\mathrm{in}(G_{n,k}))) = \left[(1-t^k)(1+t)^n\right]$.

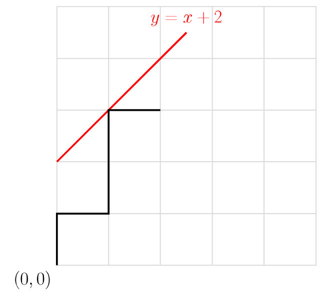
Example 6. The reduced Gröbner basis of $I_{5,2}$ is

$$G_{5,2} = \{x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, g_{\{1,2\},5,2}, g_{\{1,3,4\},5,2}, g_{\{2,3,4\},5,2}\},$$

where

$$\begin{aligned}g_{\{1,2\},5,2} &= e_2(x_1, x_2, x_3, x_4, x_5) = x_1x_2 + x_1x_3 + \cdots + x_4x_5,\\ g_{\{1,3,4\},5,2} &= e_3(x_1, x_3, x_4, x_5) = x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_3x_4x_5,\\ g_{\{2,3,4\},5,2} &= e_3(x_2, x_3, x_4, x_5) = x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5.\end{aligned}$$

The lattice path associated to $\mathrm{in}(g_{\{1,3,4\},5,2}) = x_1x_3x_4$ is illustrated below.



Enumeration and more Lefschetz

Counting the Gröbner basis elements of specified degrees gives some nice combinatorial sequences.

Proposition 7. *Fix n large and let $c_{d,k}$ be the number of $g_{A,n,k} \in G_{n,k}$ of degree d. Then $c_{d,1}$ is an enumeration of the Catalan numbers and*

$$\sum_{d=0}^{\infty} c_{d,k}t^d = \left(\sum_{d=0}^{\infty} c_{d,1}t^d\right)^k.$$

Note that $c_{d,k}$ is also the number of lattice paths that stay strictly below $y = x + k$ until it reaches the line at $(d-k, d)$ with its last north step.

Having given a new proof that $R/(x_1^2, \ldots, x_n^2)$ has the SLP, one may wonder if also $R/I_{n,k}$ has the SLP. However, this is for most interesting values of k not the case as it then fails even the weak Lefschetz property.

Theorem 8. *For general linear forms $\ell_1, \ldots, \ell_{n+1}$, the algebra $R/(\ell_1^2, \ldots, \ell_n^2, \ell_{n+1}^k) \cong R/(x_1^2, \ldots, x_n^2, (x_1 + \cdots + x_n)^k) = R/I_{n,k}$ has the weak Lefschetz property iff*

$$\begin{cases} k \geq \frac{n-3}{2} & \text{for } n \text{ odd,}\\ k \geq \frac{n}{2} & \text{for } n \text{ even.}\end{cases}$$

The proof uses the structure of $\mathrm{in}(I_{n,k})$ for the parts where it has the WLP and some explicit relations between powers of general linear forms in $R/(x_1^2, \ldots, x_n^2)$ for the failure of the WLP.

What about higher powers?

Replacing $(x_1^2, \ldots, x_n^2)$ with $(x_1^{m_1}, \ldots, x_n^{m_n})$, one can show similar results for $(x_1^{m_1}, \ldots, x_n^{m_n}, (x_1 + \cdots + x_n)^k)$. The initial ideals can in this case also be found using some lattice path bijections, and counting the number of elements of each degree can give the Motzkin, Riordan and other interesting sequences. However, the explicit Gröbner basis elements are now more complicated than just elementary symmetric polynomials. See [JKLMO25] for more details.

References

[JKLM+24] Filip Jonsson Kling, Samuel Lundqvist, Fatemeh Mohammadi, Matthias Orth, and Eduardo Sáenz-de Cabezón. Gröbner bases, resolutions, and the Lefschetz properties for powers of a general linear form in the squarefree algebra. *arXiv preprint arXiv:2411.10209*, 2024.

[JKLMO25] Filip Jonsson Kling, Samuel Lundqvist, Fatemeh Mohammadi, and Matthias Orth. The Gröbner basis for powers of a general linear form in a monomial complete intersection. *arXiv preprint arXiv:2506.24028*, 2025.