Equivariant γ-nonnegativity of order polytopes of graded posets

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Ehrhart Theory

Non-equivariant Ehrhart theory

$Q \subset \mathbb{R}^d$: lattice polytope of dimension d

$$\mathrm{Ehr}(Q;t) := 1 + \sum_{m \geq 1} |mQ \cap \mathbb{Z}^d| t^m = \frac{h^*_Q(t)}{(1-t)^{d+1}}$$

The polynomial $h^*_Q(t)$ is called the **h^*-polynomial** of Q.

- $h^*_Q(t) \in \mathbb{Z}_{\geq 0}[t]$ with degree $\leq d$
- $h^*_Q(t)$: palindromic $\iff \exists r \in \mathbb{Z}_{>0}$ s.t. rQ is reflexive

Equivariant Ehrhart theory

G : **finite group acting linearly on** $\mathbb{Z}^d$ i.e. $\exists \rho : G \to \mathrm{GL}(\mathbb{Z}^d)$
$Q \subset \mathbb{R}^d$: lattice poly. of dim d invariant under G-action
R_G : the complex ring of virtual characters of G
$\chi_{mQ} \in R_G$: the character of the representation associated with the permutation of $mQ \cap \mathbb{Z}^d$ by G for $m \in \mathbb{Z}_{>0}$

$$\mathrm{Ehr}(Q,\rho;t) := 1 + \sum_{m \geq 1} \chi_{mQ} t^m \in R_G[[t]]$$

$$\mathfrak{h}^*(Q,\rho;t) := \mathrm{Ehr}(Q,\rho;t) \cdot (1-t)\det(I - \rho t)$$

The series (sometimes, polynomial) $\mathfrak{h}^*(Q,\rho;t) \in R_G[[t]]$ is called the **equivariant h^*-series/polynomial** of Q w.r.t. G.

REMARK For $g \in G$, let $Q^g = \{x \in Q : g \cdot x = x\}$. Then

$$\mathrm{Ehr}(Q,\rho;t)(g) = \mathrm{Ehr}(Q^g;t) \quad \text{for } \forall g \in G$$

• $g(t) = \sum_{i=0}^s b_i t^i \in R_G[t]$: **effective** $\overset{\text{def}}{\iff}$ each b_i is an actual character, i.e., $b_i \in \sum_{\chi:\text{irr. char.}} \mathbb{Z}_{\geq 0}\chi$.

γ-nonnegativity

Let $f(t) = \sum_{i=0}^s a_i t^i \in \mathbb{Z}_{\geq 0}[t]$ with $a_i = a_{s-i}$ $(\forall i)$. Then $\exists \gamma_0, \ldots, \exists \gamma_{\lfloor s/2 \rfloor}$ s.t. $f(t) = \sum_{i=0}^{\lfloor s/2 \rfloor} \gamma_i t^i (1+t)^{s-2i}$.

$f(t)$: **γ-nonnegative** $\overset{\text{def}}{\iff} \gamma_i \geq 0$ $(\forall i)$

$\gamma(f;t) := \sum_{i=0}^{\lfloor s/2 \rfloor} \gamma_i t^i$: **γ-polynomial** of $f(t)$

EXAMPLE $Q = [0,1]^3$ $G = \mathfrak{S}_3 \curvearrowright \mathbb{Z}^3$

$$\Longrightarrow \mathrm{Ehr}(Q,\rho;t)(e) = \mathrm{Ehr}(Q;t) = \frac{1+4t+t^2}{(1-t)^4}$$
$$(\det(I-\rho(e)t) = (1-t)^3)$$

$$\mathrm{Ehr}(Q,\rho;t)((1\,2)) = \mathrm{Ehr}(Q^{(1\,2)};t) = \frac{1+t}{(1-t)^3} = \frac{1+2t+t^2}{(1-t)^2(1-t^2)}$$
$$(\det(I-\rho((1\,2))t) = (1-t^2)(1-t))$$

$$\mathrm{Ehr}(Q,\rho;t)((1\,2\,3)) = \mathrm{Ehr}(Q^{(1\,2\,3)};t) = \frac{1}{(1-t)^2} = \frac{1+t+t^2}{(1-t)(1-t^3)}$$
$$(\det(I-\rho((1\,2\,3))t) = 1-t^3)$$

Hence, $\mathfrak{h}^*(Q,\rho;t) = 1 + (2+\chi)t + t^2 = 1 \cdot (1+t)^2 + \chi \cdot t^1$

$\mathfrak{S}_3$	1	sgn	χ
e	1	1	2
$(1\,2)$	1	-1	0
$(1\,2\,3)$	1	1	-1

Order polytopes

$P = \{p_1, \ldots, p_d\}$: poset equipped with a partial order $\prec$

$$\mathcal{O}(P) := \left\{(x_i) \in [0,1]^d : x_i \leq x_j \text{ if } p_j \preceq p_i \text{ in } P\right\}$$

We call $\mathcal{O}(P)$ the **order polytope** of P ([3]).

P : **graded** $\overset{\text{def}}{\iff}$ every maximal saturated chain in P has the same length

Theorem ([1])
P : graded poset $\Rightarrow h^*_{\mathcal{O}(P)}(t)$ is γ-nonnegative

Main Result

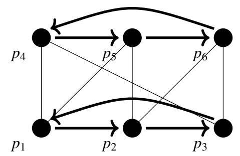
Our Goal

Find a good class of polytopes with group actions having effective γ-polynomials.

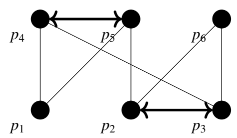
Theorem (see [2])
P : graded poset G : a subgroup of $\mathrm{Aut}(P)$
Then $\mathfrak{h}^*(\mathcal{O}(P),\rho;t)$ is γ-effective.

Example

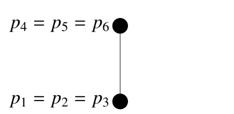
$G = D_3 (\cong \mathfrak{S}_3) = \langle \sigma, \tau : \sigma^3 = \tau^2 = e \rangle$
$P = \{p_1 \lessdot p_5 \gtrdot p_2 \lessdot p_6 \gtrdot p_3 \lessdot p_4 \gtrdot p_1\}$ $G \curvearrowright P$ as follows:



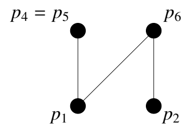
The action by σ



The action by τ



The quotient poset P/σ



The quotient poset P/τ

Then $G \curvearrowright \mathbb{Z}^6$ makes $\mathcal{O}(P)$ invariant.

$$\mathrm{Ehr}(\mathcal{O}(P),\rho;t)(e) = \mathrm{Ehr}(\mathcal{O}(P);t) = \frac{1+11t+24t^2+11t^3+t^4}{(1-t)^7}$$

$$\mathrm{Ehr}(\mathcal{O}(P),\rho;t)(\tau) = \mathrm{Ehr}(\mathcal{O}(P/\tau);t) = \frac{1+5t+8t^2+5t^3+t^4}{(1-t)^3(1-t^2)^2}$$

$$\mathrm{Ehr}(\mathcal{O}(P),\rho;t)(\sigma) = \mathrm{Ehr}(\mathcal{O}(P/\sigma);t) = \frac{1+2t+3t^2+2t^3+t^4}{(1-t)(1-t^3)^2}$$

Hence,

$$\mathfrak{h}^*(\mathcal{O}(P),\rho;t) = 1 + t^4 + (5+3\chi)(t+t^3) + (9+\mathrm{sgn}+7\chi)t^2$$
$$= 1 \cdot (1+t)^4 + (1+3\chi) \cdot t(1+t)^2 + (1+\mathrm{sgn}+\chi) \cdot t^2$$

Reference

[1] P. Brändën, Sign-graded posets, unimodality of W-polynomials and the Charney-Davis conjecture, *Electron. J. Combin.* **11** (2004).
[2] A. D'Alì and A. Higashitani, Order polytopes of graded posets are gamma-effective, arXiv:2505.07623.
[3] R. P. Stanley, Two poset polytopes, *Discrete Comput. Geom.* **1** (1986), 9–23.