Descents and Flag Major Index on Conjugacy Classes of $\mathfrak{S}_{n,r}$ without Short Cycles

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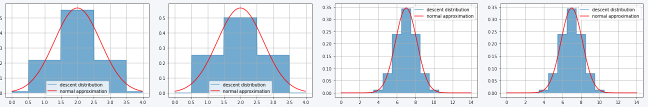
Motivation

Theorem 1 (Fulman '98). Let C_λ be a conjugacy class of $\mathfrak{S}_n$ with no cycles of lengths $1, 2, \ldots, 2k$. Then the k-th moments of the descent and major index statistics on $\mathfrak{S}_n$ align with the respective k-th moments on C_λ.

Corollary 2 (Fulman '98). Let C_{λ_n} be a conjugacy class of $\mathfrak{S}_n$ such that for all i, the number of cycles of length i in λ_n approaches 0 as $n \to \infty$. Then the distributions of the descent and major index statistics on C_{λ_n} are asymptotically the same as their distributions on $\mathfrak{S}_n$ (and hence normal).

Example

The distributions of descents on $\mathfrak{S}_5$, $C_{(5)}$, $\mathfrak{S}_{15}$, and $C_{(15)}$ are shown below.



Colored Permutation Groups

Consider r copies of the integers $\{1, 2, \ldots, n\}$, each colored by an element in $\mathbb{Z}_r$:

$$\left\{ i^{[c]} : i \in \{1, 2, \ldots, n\}, [c] \in \mathbb{Z}_r \right\}.$$

The colored permutation group $\mathfrak{S}_{n,r}$ consists of permutations on this set satisfying the condition

$$\text{if } \omega\left(i^{[0]}\right) = j^{[c]}, \text{ then } \omega\left(i^{[h]}\right) = j^{[c]+[h]} \text{ for all } [h] \in \mathbb{Z}_r.$$

When $r = 1$, $\mathfrak{S}_{n,1}$ is isomorphic to $\mathfrak{S}_n$, and when $r = 2$, $\mathfrak{S}_{n,2}$ is isomorphic to the signed symmetric (or hyperoctahedral) group B_n.

Example

A colored permutation $\omega \in \mathfrak{S}_{5,4}$ can be expressed in two-line and one-line notations by specifying the images of the elements with color $[0]$:

$$\omega = \begin{bmatrix} 1^{[0]} & 2^{[0]} & 3^{[0]} & 4^{[0]} & 5^{[0]} \\ 4^{[1]} & 5^{[3]} & 1^{[3]} & 3^{[1]} & 2^{[1]} \end{bmatrix} = [4^{[1]}5^{[3]}1^{[3]}3^{[1]}2^{[1]}].$$

A colored permutation can also be expressed in the two-line and one-line cycle notations:

$$\omega = \begin{pmatrix} 1^{[0]} & 4^{[0]} & 3^{[0]} \\ 4^{[1]} & 3^{[1]} & 1^{[3]} \end{pmatrix} \begin{pmatrix} 2^{[0]} & 5^{[0]} \\ 5^{[3]} & 2^{[1]} \end{pmatrix} = (4^{[1]}3^{[1]}1^{[3]})(5^{[3]}2^{[1]}).$$

Colored Cycle Type

In the cycle notation of $\omega \in \mathfrak{S}_{n,r}$, the *color* of a cycle is the sum of the colors that appear in the cycle (as an element in $\mathbb{Z}_r$). The *cycle type* of ω is the r-tuple of partitions $\boldsymbol{\lambda} = (\lambda^{[0]}, \lambda^{[1]}, \ldots, \lambda^{[r-1]})$ where $\lambda^{[c]}$ records cycle lengths for the cycles of color $[c]$ in ω.

Example

The cycle type of $\omega = (4^{[1]}3^{[1]}1^{[3]})(5^{[3]}2^{[1]})$ is $\boldsymbol{\lambda} = ((2), (3), \emptyset, \emptyset)$.

Conjugacy Classes

Fact. Two elements in $\mathfrak{S}_{n,r}$ are in the same conjugacy class if and only if they share the same cycle type.

Notation. For any r-tuple of partitions $\boldsymbol{\lambda}$ of n, $C_{\boldsymbol{\lambda}}$ denotes the conjugacy class consisting of colored permutations in $\mathfrak{S}_{n,r}$ with cycle type $\boldsymbol{\lambda}$.

Colored Permutation Statistics

The *descent set* of $\omega \in \mathfrak{S}_{n,r}$ is

$$\mathrm{Des}_{n,r}(\omega) = \left\{ i \in \{1, 2, \ldots, n\} : \omega\left(i^{[0]}\right) > \omega\left((i+1)^{[0]}\right) \right\}$$

where $>$ is with respect to the ordering

$$1^{[0]} < 2^{[0]} < 3^{[0]} < \cdots < 1^{[1]} < 2^{[1]} < 3^{[1]} < \cdots < 1^{[r-1]} < 2^{[r-1]} < 3^{[r-1]} < \ldots$$

and the convention that $(n+1)^{[0]}$ is a fixed point.

- The *descent statistic* is $\mathrm{des}_{n,r}(\omega) = |\,\mathrm{Des}_{n,r}(\omega)|$.
- The *major index statistic* is $\mathrm{maj}_{n,r}(\omega) = \sum_{i \in \mathrm{Des}_{n,r}(\omega) \cap [n-1]} i$.
- The *color statistic* $\mathrm{col}_{n,r}(\omega)$ is the sum (in $\mathbb{Z}$) of the colors in the one-line notation.
- The *flag major index statistic* is $\mathrm{fmaj}_{n,r}(\omega) = r \cdot \mathrm{maj}_{n,r}(\omega) + \mathrm{col}_{n,r}(\omega)$

Example

The descent set of

$$\omega = \begin{bmatrix} 1^{[0]} & 2^{[0]} & 3^{[0]} & 4^{[0]} & 5^{[0]} & 6^{[0]} & 7^{[0]} & 8^{[0]} \\ 3^{[1]} & 8^{[0]} & 5^{[0]} & 6^{[1]} & 2^{[2]} & 1^{[2]} & 4^{[0]} & 7^{[1]} \end{bmatrix} = [3^{[1]}8^{[0]}5^{[0]}6^{[1]}2^{[2]}1^{[2]}4^{[0]}7^{[1]}] \in \mathfrak{S}_{8,3}$$

is $\{1, 2, 5, 6, 8\}$. For the statistics above, we find

$$\mathrm{des}_{8,3}(\omega) = 5, \mathrm{maj}_{8,3}(\omega) = 14, \mathrm{col}_{8,3}(\omega) = 7, \text{ and } \mathrm{fmaj}_{8,3}(\omega) = 3 \cdot 14 + 7 = 49.$$

Classical Permutation Statistics

When $r = 1$, $\mathrm{des}_{n,1}$ and $\mathrm{fmaj}_{n,1}$ reduce to the descent and major index statistics on $\mathfrak{S}_n$.

Known Asymptotical Distributions on $\mathfrak{S}_{n,r}$

Theorem 3 (Chow & Mansour '12). The distribution of $\mathrm{des}_{n,r}$ has mean $\mu_{n,r} = \frac{rn+r-2}{2r}$, has variance $\sigma^2_{n,r} = \frac{n+1}{12}$, and is asymptotically normal.

Theorem 4 (Chow & Mansour '12). The distribution of $\mathrm{fmaj}_{n,r}$ has mean $\mu_{n,r} = \frac{n(rn+r-2)}{4}$, has variance $\sigma^2_{n,r} = \frac{2r^2n^3+3r^2n^2+(r^2-6)n}{72}$, and is asymptotically normal.

Main Results

Theorem 5 (Liu & Yin '25+). Let $C_{\boldsymbol{\lambda}}$ be a conjugacy class of $\mathfrak{S}_{n,r}$ with no cycles of lengths $1, 2, \ldots, 2k$. Then the k-th moments of $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$ on $\mathfrak{S}_n$ align with the respective k-th moments on $C_{\boldsymbol{\lambda}}$.

Corollary 6 (Liu & Yin '25+). Let $C_{\boldsymbol{\lambda}_n}$ be a conjugacy class of $\mathfrak{S}_{n,r}$ such that for all i, the number of cycles of length i (of any color) in $\boldsymbol{\lambda}_n$ approaches 0 as $n \to \infty$. Then the distributions of $\mathrm{des}_{n,r}$ and $\mathrm{fmaj}_{n,r}$ on $C_{\boldsymbol{\lambda}_n}$ are asymptotically the same as their distributions on $\mathfrak{S}_{n,r}$ (and hence normal).

References

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Approach for Descents

Define $X_i : \mathfrak{S}_{n,r} \to \mathbb{R}$ to be the indicator function for a descent at position i,

$$X_i(\omega) = \begin{cases} 1 & \text{if } i \in \mathrm{Des}_{n,r}(\omega) \\ 0 & \text{otherwise.} \end{cases}$$

Express $\mathrm{des}_{n,r} = \sum_{i=1}^n X_i$ so that

$$\mathrm{des}^k_{n,r} = \sum_{a_1, \ldots, a_k \in \{1, 2, \ldots, n\}} X_{a_1} \cdots X_{a_k}.$$

Proving Theorem 5 for $\mathrm{des}_{n,r}$ reduces to showing that when $\boldsymbol{\lambda}$ has no cycles of lengths $1, 2, \ldots, 2k$,

$$\mathbb{E}[X_{a_1} \cdots X_{a_k}] = \mathbb{E}[X_{a_1} \cdots X_{a_k} \mid C_{\boldsymbol{\lambda}}].$$

Proof by Example: Expectation of Descents on $\mathfrak{S}_{n,r}$

The statistic $X_1X_5X_6$ on $\mathfrak{S}_{7,3}$ takes value 1 on permutations

$$\left[\begin{array}{cc|c|c|ccc} 1^{[0]} & 2^{[0]} & 3^{[0]} & 4^{[0]} & 5^{[0]} & 6^{[0]} & 7^{[0]} \\ i_1^{[c_1]} & i_2^{[c_2]} & i_3^{[c_3]} & i_4^{[c_4]} & i_5^{[c_5]} & i_6^{[c_6]} & i_7^{[c_7]} \end{array}\right]$$

when the images within the blocks $\mathscr{B}_1 = \{1^{[0]}, 2^{[0]}\}$, $\mathscr{B}_2 = \{3^{[0]}\}$, $\mathscr{B}_3 = \{4^{[0]}\}$, and $\mathscr{B}_4 = \{5^{[0]}, 6^{[0]}, 7^{[0]}\}$ are in decreasing order.

Define $\mathfrak{S}_{\mathscr{B}_i}$ to be the permutations on $\mathscr{B}_i$, and let $\mathfrak{S}_{\mathscr{B}_1} \times \mathfrak{S}_{\mathscr{B}_2} \times \mathfrak{S}_{\mathscr{B}_3} \times \mathfrak{S}_{\mathscr{B}_4}$ act on $\mathfrak{S}_{7,3}$ by right multiplication. One can show that each orbit

- has size $2! \cdot 1! \cdot 1! \cdot 3!$, and
- contains exactly one element satisfying the decreasing order conditions.

Consequently, $\mathbb{E}[X_1X_5X_6] = \frac{1}{2! \cdot 1! \cdot 1! \cdot 3!}$.

Proof by Example: Expectation of Descents on $C_{\boldsymbol{\lambda}}$

Consider $X_1X_5X_6$ on $C_{((7), \emptyset, \emptyset)}$, which has no cycles of lengths $1, 2, 3, 4, 5, 6$. Let $\mathfrak{S}_{\mathscr{B}_1} \times \mathfrak{S}_{\mathscr{B}_2} \times \mathfrak{S}_{\mathscr{B}_3} \times \mathfrak{S}_{\mathscr{B}_4}$ act on $C_{((7), \emptyset, \emptyset)}$ by conjugation. One orbit is shown below.

$$\begin{array}{ll}
(1^{[1]}3^{[0]}5^{[2]}6^{[0]}2^{[1]}4^{[0]}7^{[2]}) & (2^{[1]}3^{[0]}5^{[2]}6^{[0]}1^{[1]}4^{[0]}7^{[2]}) \\
(1^{[1]}3^{[0]}5^{[2]}7^{[0]}2^{[1]}4^{[0]}6^{[2]}) & (2^{[1]}3^{[0]}5^{[2]}7^{[0]}1^{[1]}4^{[0]}6^{[2]}) \\
(1^{[1]}3^{[0]}6^{[2]}5^{[0]}2^{[1]}4^{[0]}7^{[2]}) & (2^{[1]}3^{[0]}6^{[2]}5^{[0]}1^{[1]}4^{[0]}7^{[2]}) \\
(1^{[1]}3^{[0]}6^{[2]}7^{[0]}2^{[1]}4^{[0]}5^{[2]}) & (2^{[1]}3^{[0]}6^{[2]}7^{[0]}1^{[1]}4^{[0]}5^{[2]}) \\
(1^{[1]}3^{[0]}7^{[2]}5^{[0]}2^{[1]}4^{[0]}6^{[2]}) & (2^{[1]}3^{[0]}7^{[2]}5^{[0]}1^{[1]}4^{[0]}6^{[2]}) \\
(1^{[1]}3^{[0]}7^{[2]}6^{[0]}2^{[1]}4^{[0]}5^{[2]}) & (2^{[1]}3^{[0]}7^{[2]}6^{[0]}1^{[1]}4^{[0]}5^{[2]})
\end{array}$$

One can show that each orbit

- has size $2! \cdot 1! \cdot 1! \cdot 3!$, and
- contains exactly one element satisfying the decreasing order conditions.

Consequently, $\mathbb{E}[X_1X_5X_6 \mid C_{((7), \emptyset, \emptyset)}] = \frac{1}{2! \cdot 1! \cdot 1! \cdot 3!}$.

Key Lemma for Descents

Lemma 7. Suppose $C_{\boldsymbol{\lambda}}$ contains no cycles of lengths $1, 2, \ldots, 2k$. Let $\mathscr{B}_1, \ldots, \mathscr{B}_t$ be the blocks induced by the decreasing conditions needed for descents at $a_1, \ldots, a_k$, where $\mathscr{B}_t$ contains n. If $n \notin \{a_1, \ldots, a_k\}$, then

$$\mathbb{E}[X_{a_1} \ldots X_{a_k}] = \frac{1}{\prod_{j=1}^t |\mathscr{B}_j|!} = \mathbb{E}[X_{a_1} \ldots X_{a_k} \mid C_{\boldsymbol{\lambda}}],$$

If $n \in \{a_1, \ldots, a_k\}$, then

$$\mathbb{E}[X_{a_1} \ldots X_{a_k}] = \left(\frac{r-1}{r}\right)^{|\mathscr{B}_t|} \cdot \frac{1}{\prod_{j=1}^t |\mathscr{B}_j|!} = \mathbb{E}[X_{a_1} \ldots X_{a_k} \mid C_{\boldsymbol{\lambda}}],$$