On some Grothendieck expansions

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Setting and motivation

Let $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ with *Borel subgroup* B of lower-triangular matrices.

Let S_n be the *symmetric group* on $\{1, \dots, n\}$, and set $s_i = (i, i+1) \in S_n$.

- The *complete flag variety* GL_n/B has a decomposition into *Schubert cells* BwB/B. The *Schubert classes* $[X_w] = [\overline{Bw_0wB/B}]$ $(w \in S_n)$ form a basis for $H^*(\mathrm{GL}_n/B)$.
- A subgroup $H \le \mathrm{GL}_n$ is *spherical* if H acts on GL_n/B with finitely many orbits.

  Brion (2001) proved that if H is spherical and $\mathcal{O} \subset \mathrm{GL}_n/B$ is an H-orbit, then
  $$[\overline{\mathcal{O}}] = \textstyle\sum_{w \in \mathcal{A}(\mathcal{O})} 2^{d_{\mathcal{O}}(w)} [X_w] \in H^*(\mathrm{GL}_n/B)$$
  where $\mathcal{A}(\mathcal{O}) \subseteq S_n$ and $d_{\mathcal{O}}(w) \in \mathbb{Z}_{\ge 0}$ are defined via an explicit *weak order graph*.
- We want to find a K-theory analog when $H = \mathrm{O}_n := \{g \in \mathrm{GL}_n \mid g^{-1} = g^\dagger\}$.

Demazure product and orthogonal orbits

The *Demazure product* is the unique associative product $\circ : S_n \times S_n \to S_n$ such that
$$w \circ s_i = ws_i \text{ if } w(i) < w(i+1) \quad \text{and} \quad w \circ s_i = w \text{ if } w(i) > w(i+1).$$

O_n-orbits in GL_n/B are indexed by $I_n := \{z \in S_n \mid z = z^{-1}\}$ and form a digraph:

- the unique source is the orbit indexed by $z = 1$,
- edges are $\mathcal{O}_z \xrightarrow{s_i} \mathcal{O}_{s_i \circ z \circ s_i}$ if $z(i) < z(i+1)$, doubled if $i = z(i)$ & $i+1 = z(i+1)$.

In Brion's formula $\mathcal{A}(\mathcal{O}_z)$ is set of $w = s_{i_1} \cdots s_{i_k}$ arising from paths $\mathcal{O}_1 \xrightarrow{s_{i_1}} \cdots \xrightarrow{s_{i_k}} \mathcal{O}_z$.

The integer $d_{\mathcal{O}_z}(w)$ is the number of doubled edges in any such path.

Algebraic K-theory and Grothendieck polynomials

We consider the *algebraic K-theory of coherent sheaves* on GL_n/B.

- Lascoux (1990): for $w \in S_n$, the *Grothendieck polynomials* $\mathcal{G}_w \in \mathbb{Z}[x_1, \cdots, x_n]$ are computed inductively with the base case $\mathcal{G}_{n\cdots 321} = x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ and a certain *K-divided difference formula* $\partial_i^K \mathcal{G}_w = \mathcal{G}_{ws_i}$ when $w(i) > w(i+1)$.
- The (K-theory) Schubert classes $[X_w] := [i_*\mathcal{O}_{X_w}]$ form a basis for $K(\mathrm{GL}_n/B)$. Each $\mathcal{G}_w$ represents $[X_w] \in K(\mathrm{GL}_n/B) \cong \mathbb{Z}[x_1, \cdots, x_n]/I\Lambda_n$.

M.–Pawlowksi introduced *orthogonal Grothendieck polynomials* $\mathcal{G}^{\mathsf{O}}_z \in \mathbb{Z}[x_1, \cdots, x_n]$ that represent the K-theory classes of the O_n-orbit closures $\overline{\mathcal{O}_z}$ for $z \in I_n$.

- No general algebraic formula for $\mathcal{G}^{\mathsf{O}}_z$ except when z is *vexillary* (2143-avoiding):

Theorem [M.–W. (2024)]

The orthogonal Grothendieck polynomials $\mathcal{G}^{\mathsf{O}}_z$ for $z \in I_n^{\mathrm{vex}}$ satisfy
$$\mathcal{G}^{\mathsf{O}}_{n\cdots 321} = \prod_{1 \le i \le j \le n-i} (x_i + x_j - x_ix_j) \text{ and } \partial_i^K \mathcal{G}^{\mathsf{O}}_{s_izs_i} = \mathcal{G}^{\mathsf{O}}_z \text{ if } \begin{cases} \ell(z) < \ell(s_izs_i) \text{ and} \\ s_izs_i \in I_n^{\mathrm{vex}}. \end{cases}$$

Results of Brion imply that for each $z \in I_n^{\mathrm{vex}}$ there is a map $\mathrm{GC}^{\mathsf{O}}_z : S_{n+1} \to \mathbb{Z}_{\ge 0}$ with
$$\mathcal{G}^{\mathsf{O}}_z = \textstyle\sum_{w \in S_{n+1}} (-1)^{\ell(w) - \ell_{\mathsf{inv}}(z)} \cdot \mathrm{GC}^{\mathsf{O}}_z(w) \cdot \mathcal{G}_w.$$

Our main goal: understand the function $\mathrm{GC}^{\mathsf{O}}_z$, which can be very complicated.

Involution Grothendieck polynomials

- Knutson (2009): If $\mathcal{O}$ is multiplicity-free, then $[\overline{\mathcal{O}}] = \sum_{w \in \mathcal{P}} \mu_w [X_w] \in K(\mathrm{GL}_n/B)$.
- The O_n-orbits are not multiplicity-free except when $z = 1$, but the right-hand side of Knutson's formula still has nice properties and approximates $\mathcal{G}^{\mathsf{O}}_z$ when z is vexillary.

Let $\mathcal{B}_{\mathsf{inv}}(z) := \{w \in S_n \mid w^{-1} \circ w = z\}$ and $\ell_{\mathsf{inv}}(z) := \min\{\ell(w) \mid w \in \mathcal{B}_{\mathsf{inv}}(z)\}$.

- The set $\mathcal{B}_{\mathsf{inv}}(z)$ consists of inverses of a single equivalence class for the relation with
  $$\cdots cba \cdots \sim \cdots cab \cdots \sim \cdots bca \cdots \quad \text{when } a < b < c.$$
- For $z \in I_n$ define $\widehat{\mathcal{G}}_z := \sum_{w \in \mathcal{B}_{\mathsf{inv}}(z)} (-1)^{\ell(w) - \ell_{\mathsf{inv}}(z)} \mathcal{G}_w$.

Theorem [M.–W. (2024)]

The *involution Grothendieck polynomials* $\widehat{\mathcal{G}}_z$ for $z \in I_n$ have the inductive formula
$$\widehat{\mathcal{G}}_{n\cdots 321} = \prod_{1 \le i \le n-i} x_i \prod_{1 \le i < j < n-i} (x_i + x_j - x_ix_j) \text{ and } \partial_i^K \widehat{\mathcal{G}}_{s_i \circ z \circ s_i} = \widehat{\mathcal{G}}_z \text{ if } z(i) < z(i+1).$$

Main Theorem [M., W. (2024)]

If $z \in I_n^{\mathrm{vex}}$ then there are explicit polynomials $\theta_{yz} \in \mathbb{Z}[1 - x_i \mid i < z(i)]$
$$\mathcal{G}^{\mathsf{O}}_z = \sum_{y \in I_n} (-1)^{\ell_{\mathsf{inv}}(y) - \ell_{\mathsf{inv}}(z)} \cdot \theta_{yz} \cdot \widehat{\mathcal{G}}_y.$$
We have $\theta_{yz} \in \mathbb{Z}_{\ge 0}[1 - x_i \mid i < z(i)]$ if it never holds that $i < z(i)$ and $i+1 < z(i+1)$.

Pieri chains and conjectural support

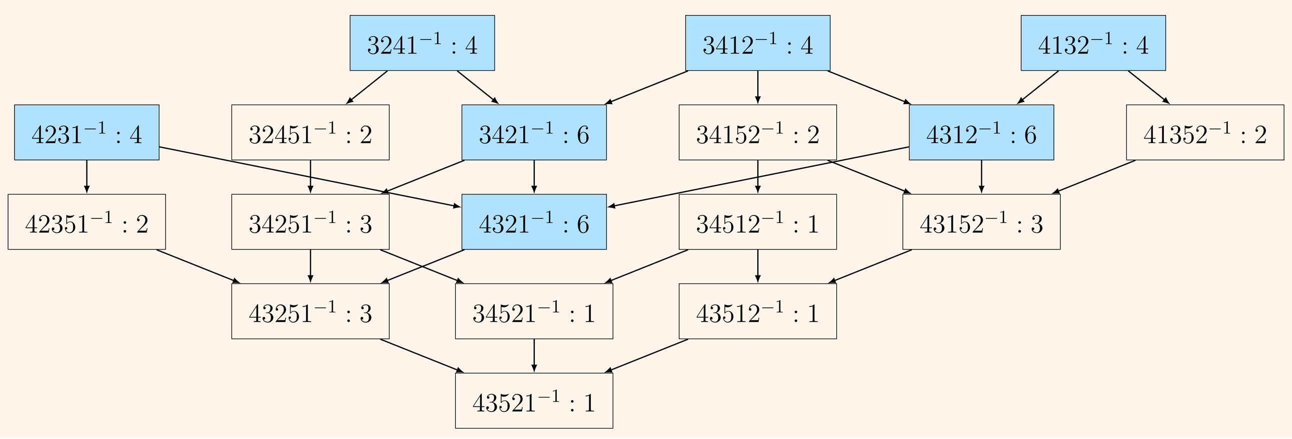
- A *k-Pieri chain* from $v \in S_n$ to $w \in S_n$ is a sequence
  $$v = v_0 \xrightarrow{(a_1,b_1)} v_1 \xrightarrow{(a_2,b_2)} \cdots \xrightarrow{(a_q,b_q)} v_q = w$$
  satisfying $1 \le a_i \le k < b_i$ for all i, along with the conditions
  $$b_1 \ge b_2 \ge \cdots \ge b_q \quad \text{and} \quad j < i \text{ and } a_j = a_i > a_{i+1} \implies b_i > b_{i+1}.$$
- Lenart–Sottile (2006): at most one such chain exists between any v and w. We write $v \xrightarrow{[k]} w$ if this occurs.

For $z \in I_n$ let $k_z = |\{i \mid i < z(i)\}|$ and
$$\mathcal{B}^+_{\mathsf{inv}}(z) := \{w \in S_{n+1} \mid v \xrightarrow{[k_z]} w \text{ for some } v \in \mathcal{B}_{\mathsf{inv}}(z)\}.$$

Thm (M.–W.): If $z \in I_n^{\mathrm{vex}}$ has $i < z(i)$ for all $i \le k_z$ then $\mathrm{supp}(\mathrm{GC}^{\mathsf{O}}_z) \subseteq \mathcal{B}^+_{\mathsf{inv}}(z)$.

Conjecture: $\mathrm{supp}(\mathrm{GC}^{\mathsf{O}}_{w_0}) = \mathcal{B}^+_{\mathsf{inv}}(w_0)$ for the longest element $w_0 = n \cdots 321 \in I_n$.

Example shows elements of $\mathcal{B}_{\mathsf{inv}}(w_0)$ and $\mathcal{B}^+_{\mathsf{inv}}(w_0)$ for $w_0 = 4321$, and values of $\mathrm{GC}^{\mathsf{O}}_{w_0}$.



Grothendieck polynomials and stable limits

For $n \in \mathbb{Z}_{\ge 0}$ and $w \in S_\infty$, let $1^n \times w(i) := w(i-n) + n$ if $i > n$, and i otherwise.

- Fomin–Kirillov (1994): the *stable limit* $G_w := \lim_{n\to\infty} \mathcal{G}_{1^n \times w}$ is symmetric.
- Buch (2002): $\{G_w : w \text{ is vexillary}\}$ is a $\mathbb{Z}$-basis for the ring $\mathbb{Z}$-span$\{G_w : w \in S_\infty\}$.

If $z \in I_n$ then define $GP_z := \lim_{n\to\infty} \widehat{\mathcal{G}}_{1^n \times z} = \sum_{w \in \mathcal{B}_{\mathsf{inv}}(z)} (-1)^{\ell(w) - \ell_{\mathsf{inv}}(z)} G_w$.

- M. (2020): $\{GP_z : z \in I_\infty^{\mathrm{vex}}\}$ is a $\mathbb{Z}$-basis for the ring $\mathbb{Z}$-span$\{GP_z : z \in I_\infty\}$ and recovers Ikeda–Naruse's *K-theoretic Schur P-functions*.

Orthogonal stable limits

For $z \in I_n^{\mathrm{vex}}$ define $GQ_z := \lim_{n\to\infty} \mathcal{G}^{\mathsf{O}}_{1^n \times z}$.

Thm (M.–W.). If $z(1) = 1$ then $GQ_z = \sum_{w \in S_{n+1}} (-1)^{\ell(w) - \ell_{\mathsf{inv}}(z)} \mathrm{GC}^{\mathsf{O}}_z(w) G_w$.

- Lewis–M. (2021): the set $\{GQ_z \mid z \in I_\infty^{\mathrm{vex}}\}$ is a $\mathbb{Z}$-basis for the ring it generates.

It is open to find a formula for the multiplicative structure constants of this ring.

- M.–Pawlowski (2020): $\{GQ_z \mid z \in I_n^{\mathrm{vex}}\}$ recovers Ikeda–Naruse's GQ-functions.
- Taking the stable limit of our main theorem recovers an identity of Chiu–M. (2023):
  $$GQ_\lambda = \textstyle\sum_\nu (-1)^{\#\mathsf{columns}(\nu/\lambda)} \cdot 2^{\ell(\lambda) - |\nu/\lambda|} \cdot GP_\nu.$$

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