# Arborescences of random covering graphs

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## Matrix-Tree Theorem

### Definition of arborescence

An *arborescence* $T$ of a weighted digraph $\Gamma$ rooted at $v \in V$ is a spanning tree directed towards $v$. The *weight* of an arborescence is the product of the weights of its edges. Define $A_v(\Gamma)$ to be the sum of weights of all arborescences rooted at $v$.

The Matrix-Tree Theorem [1] relates the minors of the Laplacian and the arborescences.

### Matrix-Tree Theorem

Let $G$ be a weighted digraph with vertex set $\{1, \cdots, n\}$. Then $A_i(G) = \det(L(G)_i^i)$ for all $i \in \{1, \cdots, n\}$. Here $L(G)_i^i$ is the Laplacian matrix $L(G)$ with row $i$ and column $i$ removed.

## Covering Graphs

A *$k$-fold cover* of a weighted digraph $\Gamma = (V, E, \mathrm{wt})$ is a weighted digraph $\tilde{\Gamma} = (\tilde{V}, \tilde{E}, \mathrm{wt})$ that is a $k$-fold covering space of $G$ in the topological sense that preserves edge weight. The following characterization of $\tilde{\Gamma}$ uses permutation-voltage $\nu(e) = \sigma_e \in \mathfrak{S}_k$ associated to each edge $e$ .

- A vertex set $\tilde{V} = V \times \{1, 2, \ldots, k\}$
- An edge set $\tilde{E} := \{(v \times x, w \times \sigma_e(x)) : x \in \{1, \ldots, k\}, e = (v, w) \in E\}$.
- A weight-preserving projection map.

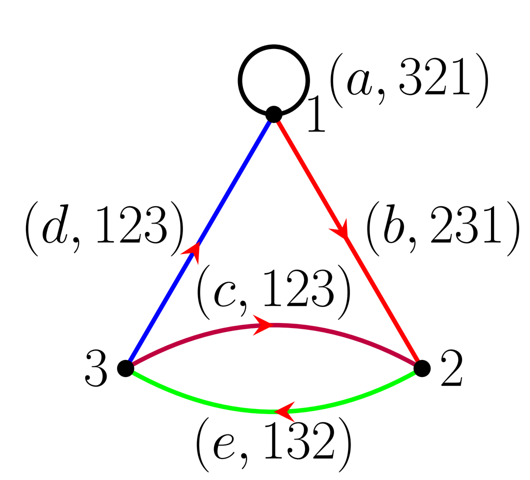


Figure 1: A permutation-voltage graph $\Gamma$.

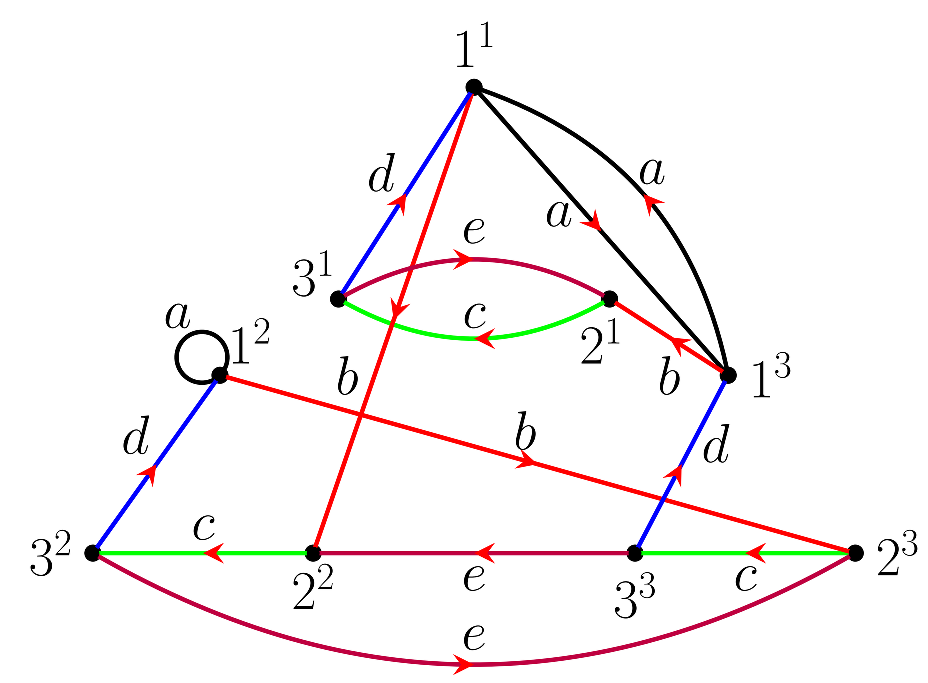

Figure 2: The derived covering graph $\tilde{\Gamma}$ of $\Gamma$ in Figure 1. Edge colors denote correspondence to the edges of $\Gamma$ via the projection map.

In fact, it is sufficient to construct a covering graph by its permutation voltage due to [2].

### Characterization of covering graphs

Every covering graph of $\Gamma$ can be constructed from a permutation voltage graph $\Gamma$.

## Arborescence ratio

Let $\tilde{\Gamma}$ be a $k$-fold cover of $\Gamma$ and $\tilde{v}$ be a lift vertex of $v$. The arborescence ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ has an explicit formula using the voltage laplacian in [3].

### Arborescence ratio formula

Denote by $\mathcal{L}(\Gamma)$ the voltage Laplacian of $\Gamma$. Then for any vertex $v$ of $\Gamma$ and any lift $\tilde{v}$ of $v$ in $\tilde{\Gamma}$, we have

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{k}\det[\mathcal{L}(\tilde{\Gamma})].$$

If the edge weights of $\Gamma$ are in indeterminates, then the polynomial $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ has integer coefficients.

## Expected value of arborescence ratio

We choose a permutation voltage $\sigma_e$ in the symmetric group $\mathfrak{S}_k$ for each edge $e \in E$ independently and uniformly at random. The random voltages allow us to construct our random $k$-fold covering graph $\tilde{\Gamma}$. [3] conjectured the expected value of arborescence ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ and we proved it.

### Expected value

$$\mathbb{E}\left[\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}\right] = \frac{1}{k}\prod_{w \in V}\left(\sum_{\alpha \in E_s(w)} \mathrm{wt}(\alpha)\right)^{k-1}$$

where $E_s(w)$ is the set of edges in $\Gamma$ with source $w$.

## Positivity Conjecture

[3] Conjectured that the coefficients of $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ are nonnegative and proved the case $k = 2$ by revealing its combinatorial interpretation. The definition of *negative vector field* can be found in [3].

### Arborescence ratio of 2-fold covering graph

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{2}\sum_{\gamma \in \mathcal{N}(\Gamma)} 2^{\#C(\gamma)}\mathrm{wt}(\gamma)$$

where $\mathcal{N}(\Gamma)$ denotes the set of negative vector fields in $\Gamma$, and $C(\gamma)$ denote the set of cycles in a vector field $\gamma$.

A direct corollary shows that for a $k$-fold cover, the $\mathrm{wt}(\gamma)^{k-1}$-coefficient of $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ is $k^{\#C(\gamma)}$ or 0. However, for general terms, the combinatorial interpretation is not clear.

## Future Directions

For $k \geq 3$, those problems remain open:

- How to prove the Positivity Conjecture?
- Is there any combinatorial interpretation for the coefficients of the arborescence ratio?

## References

[1] Sergei Fomin and Richard Stanley. *Enumerative Combinatorics Volume 2*. Cambridge University Press, 1999.

[2] Jonathan Gross and Thomas Tucker. Generating all graph coverings by permutation voltage assignments. *Discrete Mathematics*, pages 273–283, August 1975.

[3] Sunita Chepuri et al. Arborescences of covering graphs. *Algebraic Combinatorics*, 5(2):319–346, 2022.

[4] Pavel Galashin and Pavlo Pylyavskyy. $r$-systems. *Selecta Mathematica (New Series)*, 25(2), 2019.

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