Newell-Littlewood Saturation

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Littlewood-Richardson Coefficients

The irreducible polynomial representations V_λ of $GL_n\mathbb{C}$ are indexed by the set of partitions

$$\mathrm{Par}_n := \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}. \qquad (1)$$

For each $\mu, \nu \in \mathrm{Par}_n$,

$$V_\mu \otimes V_\nu \cong \bigoplus_{\lambda \in \mathrm{Par}_n} V_\lambda^{\oplus c_{\mu,\nu}^{\lambda}}. \qquad (2)$$

The tensor product multiplicities $c_{\mu,\nu}^{\lambda}$ are the **Littlewood-Richardson(LR) coefficients**.

LR Saturation Theorem

Let $\lambda, \mu, \nu \in \mathrm{Par}_n$. Then

$$\exists k \in \mathbb{N} \text{ such that } c_{k\mu,k\nu}^{k\lambda} > 0 \quad \Rightarrow \quad c_{\mu,\nu}^{\lambda} > 0. \qquad (3)$$

This is **LR saturation** proved by A. Knutson and T. Tao [7]. They used **honeycombs**, which are combinatorial objects counting LR coefficients.

Example: Honeycombs

In terms of honeycombs, LR saturation can be written as follows: If red edges have integer length, is it possible to modify black edges so that all edges have integer length? The answer is yes, according to [7].

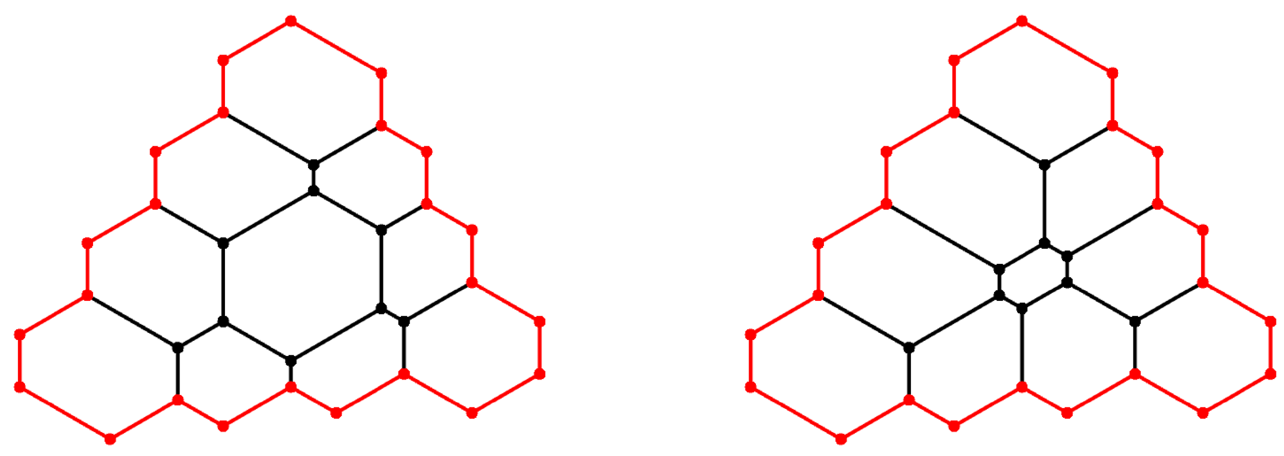

Figure: Honeycombs: Modify black edges while fixing red ones.

Consequences of LR Saturation

The significance of the saturation theorem stems from **Horn's conjecture** [4] which gives a recursive description of linear inequalities, called **Horn's inequalities**, on the eigenvalues of $n \times n$ Hermitian matrices A, B and $A + B$. LR saturation combined with earlier work of A. A. Klyachko [6] proved Horn's conjecture.

Newell-Littlewood Numbers

Define **Newell-Littlewood(NL) numbers**

$$N_{\lambda,\mu,\nu} := \sum_{\alpha,\beta,\gamma \in \mathrm{Par}_n} c_{\beta,\gamma}^{\lambda} c_{\gamma,\alpha}^{\mu} c_{\alpha,\beta}^{\nu} \quad (\lambda, \mu, \nu \in \mathrm{Par}_n). \qquad (4)$$

For each $\lambda \in \mathrm{Par}_n$, let $|\lambda| := \lambda_1 + \cdots + \lambda_n$. If $c_{\mu,\nu}^{\lambda} \neq 0$, then $|\mu| + |\nu| = |\lambda|$. According to [1, Lemma 2.2],

$$|\mu| + |\nu| = |\lambda| \quad \Rightarrow \quad N_{\lambda,\mu,\nu} = c_{\mu,\nu}^{\lambda}. \qquad (5)$$

Thus, NL numbers generalize LR coefficients.

Let $G = SO_{2n+1}\mathbb{C}$, $Sp_{2n}\mathbb{C}$, $SO_{2n}\mathbb{C}$. Write $c_{\mu,\nu}^{\lambda}(G)$ as tensor product multiplicities with respect to G. $l(\lambda)$ denotes the number of non-zero components of $\lambda = (\lambda_1, \cdots, \lambda_n)$. According to [8, Theorem 3.1],

$$l(\mu) + l(\nu) \leqslant n \quad \Rightarrow \quad N_{\lambda,\mu,\nu} = c_{\mu,\nu}^{\lambda}(G). \qquad (6)$$

The condition imposed on $\mu, \nu \in \mathrm{Par}_n$ is called the **stable range**.

Main Theorem (NL Saturation)

Let $\lambda, \mu, \nu \in \mathrm{Par}_n$ satisfying $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$. Then

$$\exists k \in \mathbb{N} \text{ such that } N_{k\lambda,k\mu,k\nu} > 0 \quad \Rightarrow \quad N_{\lambda,\mu,\nu} > 0. \qquad (7)$$

This is **Newell-Littlewood saturation** proved by M. in [9]. M. used **Möbius honeycombs**, which are combinatorial objects counting NL numbers.

Example: Möbius Honeycombs

In terms of Möbius honeycombs, NL saturation can be restated as follows: If red edges have integer lengths on a Möbius strip, is it possible to modify black edges so that all edges have integer length?

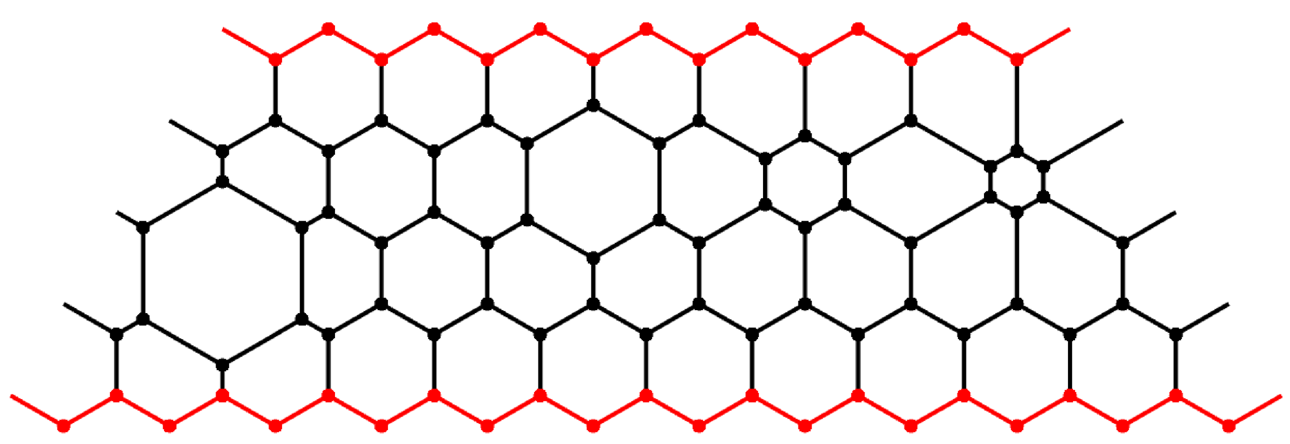

Figure: Möbius honeycombs on a Möbius strip: Modify black edges while fixing red ones.

Consequences of Main Theorem

Analogous to the Horn's inequalities, S. Gao, G. Orelowitz and A. Yong [2, Theorem 1.3] defined **extended Horn inequalities** and proved that they are necessary conditions for $N_{\lambda,\mu,\nu} > 0$. Additionally, they conjectured the converse; The Main Theorem confirms this conjecture, due to [3, Corollary 8.5].

Secondly, combined with [3, Proposition 3.1] proved by S. Gao, G. Orelowitz, N. Ressayre and A. Yong, we complete an analogue of the Horn problem for matrices in $\mathfrak{sp}_{2n}\mathbb{C} \cap \mathfrak{u}_{2n}\mathbb{C}$.

Lastly, let $G = SO_{2n+1}\mathbb{C}$, $Sp_{2n}\mathbb{C}$, $SO_{2n}\mathbb{C}$. Suppose $\lambda, \mu, \nu \in \mathrm{Par}_n$ and $l(\mu) + l(\nu) \leqslant n$. Then as a corollary, we have

$$\exists k \in \mathbb{N} \text{ such that } c_{k\mu,k\nu}^{k\lambda}(G) > 0 \quad \Rightarrow \quad c_{\mu,\nu}^{\lambda}(G) > 0. \qquad (8)$$

This gives partial answer to [5, Conjecture 1.4] and [7, Section 7].

Sketch of Proof

Unlike LR saturation, NL saturation needs **parity condition** $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$. Surprisingly, this is related to the fact that Möbius strip may have non-orientable loops.

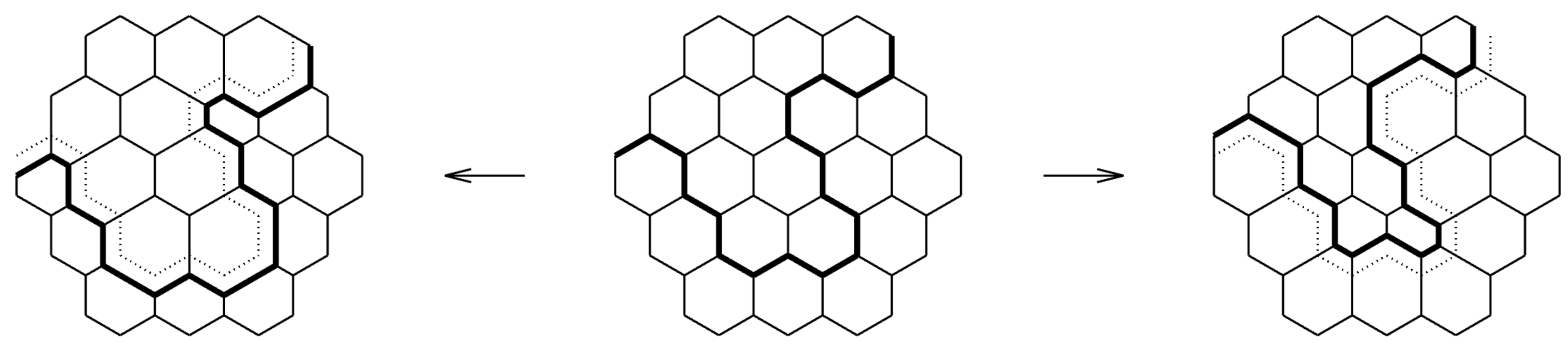

Figure: Sliding an orientable loop.

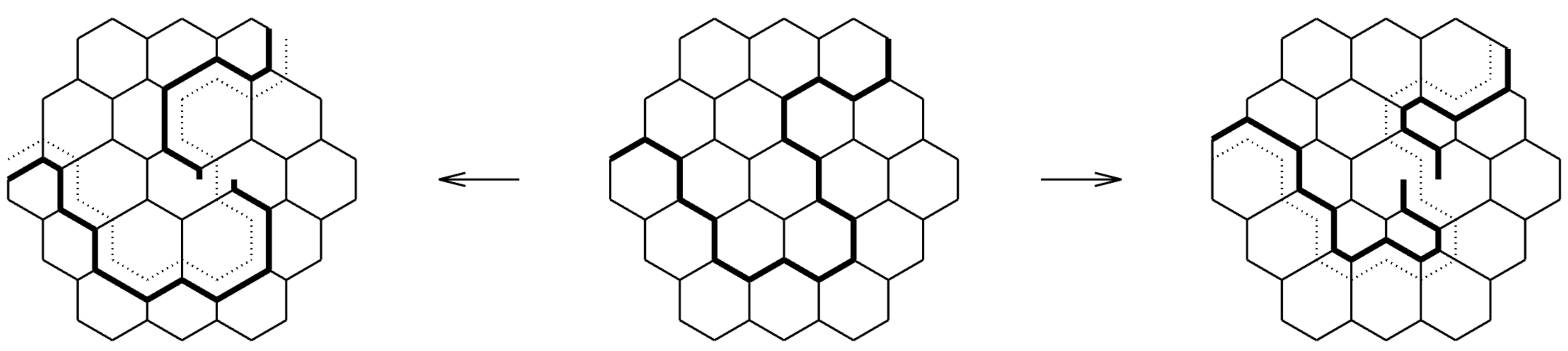

Figure: Breaking a non-orientable loop.

Example: Counting NL Numbers

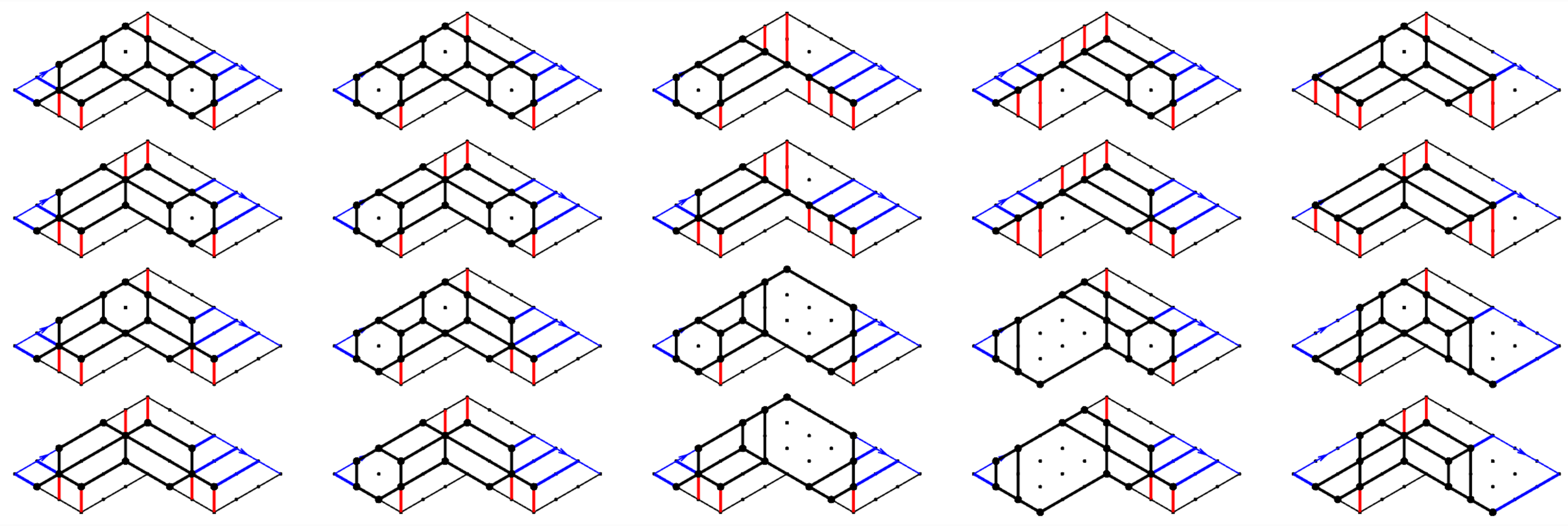

Figure: Möbius honeycombs corresponding to $\lambda = \mu = \nu = (3, 2, 1)$. Then $N_{\lambda,\mu,\nu} = 20$.

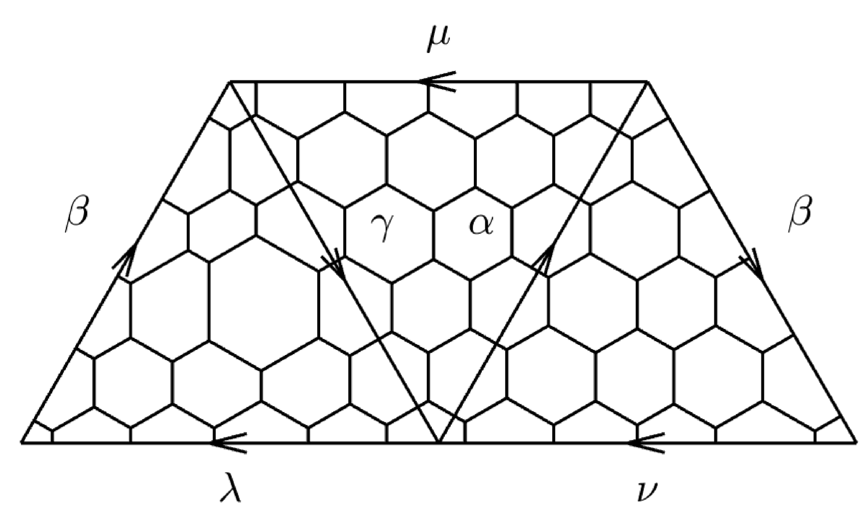

Figure: Combining three honeycombs to construct a Möbius honeycomb on a Möbius strip.

References

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