Variations of the (α, t)-Eulerian polynomials and gamma positivity

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Abstract

We define a multivariable generalization of the Eulerian polynomials using linear and descent based statistics of permutations and establish the connection with the (α, t)-Eulerian polynomials based on cyclic and excedance based statistics of permutations. As applications of this connection, we obtain the exponential generating function for the multivariable Eulerian polynomials and γ-positive formulas of two variants of Eulerian polynomials. We also show that enumerating the cycle André permutations with respect to the number of drops, fixed points and cycles gives rise to the normalised γ-vectors of the (α, t)-Eulerian polynomials. Our result generalizes and unifies several recent results in the literature.

Introduction

For any positive integer n, we denote the symmetric group of $[n] := \{1, 2, \dots, n\}$ by $\mathfrak{S}_n$. For $\sigma \in \mathfrak{S}_n$, the integer $i \in [n-1]$ is called a *descent* (**des**) if $\sigma(i) > \sigma(i+1)$; an *ascent* (**asc**) if $\sigma(i) < \sigma(i+1)$; an *excedance* (**exc**) if $i < \sigma(i)$. It is well-known that the Eulerian polynomials $A_n(x)$ have the following combinatorial interpretations:

$$A_n(x) := \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{asc}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{exc}(\sigma)}. \tag{1}$$

Let $\mathcal{M}_n$ be the set of permutations $\sigma \in \mathfrak{S}_n$ such that the first descent (if any) of σ appears at $\sigma^{-1}(n)$. The *binomial-Eulerian polynomials* were introduced by Postnikov, Reiner, and Williams as the h-polynomials of stellohedrons, and can also be defined as in the following

$$\widetilde{A}_n(x) := \sum_{\sigma\in\mathcal{M}_{n+1}} x^{\mathrm{des}(\sigma)} = 1 + x\sum_{m=1}^{n}\binom{n}{m}A_m(x). \tag{2}$$

It is well-knownthat the Eulerian polynomials $A_n(x)$ have the following γ-positive expansion

$$A_{n+1}(x) = \sum_{j=0}^{\lfloor n/2\rfloor}\gamma_{n,j}\, x^j(1+x)^{n-2j} = \sum_{j=0}^{\lfloor n/2\rfloor} 2^j d_{n,j}\, x^j(1+x)^{n-2j}, \tag{3}$$

where $\gamma_{n,j}$ is the number of permutations without double descents having j descents in $\mathfrak{S}_{n+1}$ and $d_{n,j}$ is the number of André permutations with j descents in $\mathfrak{S}_{n+1}$. It is also known that the polynomials $\widetilde{A}_n(x)$ have the following gamma positive formula

$$\widetilde{A}_n(x) = \sum_{j=0}^{\lfloor n/2\rfloor}\widetilde{\gamma}_{n,j}\, x^j(1+x)^{n-2j}, \tag{4}$$

where $\widetilde{\gamma}_{n,j}$ is the number of $\sigma \in \mathcal{M}_{n+1}$ such that σ has j descents and no double descents. For $\sigma \in \mathfrak{S}_n$, an index $i \in [n]$ is a *drop* (**drop**) of σ if $i > \sigma(i)$; a *fixed point* (**fix**) of σ if $i = \sigma(i)$. We shall also consider a permutation $\sigma \in \mathfrak{S}_n$ as a word $\sigma = \sigma_1 \dots \sigma_n$ with $\sigma_i := \sigma(i)$ for $i \in [n]$. Say that a letter σ_i is a *left-to-right maximum* (**lrmax**) of σ if $\sigma_i > \sigma_j$ for every $j < i$; a *right-to-left maximum* (**rlmax**) of σ if $\sigma_i > \sigma_j$ for every $j > i$.

In the middle of 1970's Carlitz-Scoville considered several multivariate Eulerian polynomials, among which are the so-called (α, β)-Eulerian polynomials

$$A_n(x, y \,|\, \alpha, \beta) := \sum_{\sigma\in\mathfrak{S}_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} \alpha^{\mathrm{lmax}(\sigma)-1}\beta^{\mathrm{rmax}(\sigma)-1}, \tag{5a}$$

and the following (α, t)-**Eulerian polynomials**,

$$A_n^{\mathrm{cyc}}(x, y, t \,|\, \alpha) := \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} y^{\mathrm{drop}(\sigma)} t^{\mathrm{fix}(\sigma)}\alpha^{\mathrm{cyc}(\sigma)}, \tag{5b}$$

where $\mathrm{cyc}(\sigma)$ denotes the number of cycles of σ.

For a permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, we say that an index $i \in [n]$ is a *cycle peak* (**cpk**) of σ if $\sigma^{-1}(i) < i > \sigma(i)$; *cycle valley* (**cval**) of σ if $\sigma^{-1}(i) > i < \sigma(i)$; *cycle double ascent* (**cda**) of σ if $\sigma^{-1}(i) < i < \sigma(i)$; *cycle double descent* (**cdd**) of σ if $\sigma^{-1}(i) > i > \sigma(i)$. Note that $\mathrm{cpk}(\sigma) = \mathrm{cval}(\sigma)$.

Theorem 1

If $xy = u_1u_2$ and $x + y = u_3 + u_4$, then

$$A_n^{\mathrm{cyc}}(x, y, t \,|\, \alpha) = \sum_{\sigma\in\mathfrak{S}_n}(u_1u_2)^{\mathrm{cpk}(\sigma)}u_3^{\mathrm{cda}(\sigma)}u_4^{\mathrm{cdd}(\sigma)}t^{\mathrm{fix}(\sigma)}\alpha^{\mathrm{cyc}(\sigma)}. \tag{6}$$

Let $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ with the boundary condition $\mathbf{0} - \mathbf{0}$. A letter $\sigma_i \in [n]$ is a

- *left-to-right-maximum-peak* (**lmaxpk**) if σ_i is a left-to-right maximum and also a peak;
- *right-to-left-maximum-peak* (**rmaxpk**) if σ_i is a right-to-left maximum and also a peak;
- *left-to-right-maximum-double-ascent* (**lmaxda**) if σ_i is a left-to-right maximum and also a double ascent;
- *right-to-left-maximum-double-descent* (**rmaxdd**) if σ_i is a right-to-left maximum and also a double descent.

Let $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and define the generalized Eulerian polynomial

$$A_n(\mathbf{u}, f, g, t \,|\, \alpha, \beta) = \sum_{\sigma\in\mathfrak{S}_{n+1}}(u_1u_2)^{\mathrm{val}(\sigma)}u_3^{\mathrm{da}(\sigma)}u_4^{\mathrm{dd}(\sigma)}f^{\mathrm{lmaxpk}(\sigma)-1}g^{\mathrm{rmaxpk}(\sigma)-1} \times t^{\mathrm{lmaxda}(\sigma)+\mathrm{rmaxdd}(\sigma)}\alpha^{\mathrm{lmax}(\sigma)-1}\beta^{\mathrm{rmax}(\sigma)-1}.$$

The following is our second main result.

Theorem 2

If $xy = u_1u_2$ and $x + y = u_3 + u_4$, then

$$A_n(\mathbf{u}, f, g, t \,|\, \alpha, \beta) = A_n^{\mathrm{cyc}}\left(x, y, \frac{\alpha u_3 + \beta u_4}{\alpha f + \beta g}t \,\Big|\, \alpha f + \beta g\right). \tag{7}$$

Sketch of the proof

We prove Theorem 1 by combining a variant of Foata's fundamental transformation with cyclic valley hopping and Theorem 2 by establishing a bijection.

A symmetric (α, t)-Eulerian identity

Define two kinds of (α, t)-*Eulerian numbers* as follows:

$$\left\langle {n \atop k} \right\rangle^{\mathrm{exc}}_{\alpha,t} := \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{exc}(\sigma)=n-k}} \alpha^{\mathrm{cyc}(\sigma)}t^{\mathrm{fix}(\sigma)} \qquad (1 \le k \le n), \tag{8a}$$

and

$$\left\langle {n \atop k} \right\rangle^{\mathrm{asc}}_{\alpha,t} := \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{asc}(\sigma)=n-k}} \alpha^{\mathrm{rmax}(\sigma)}t^{\mathrm{rmaxdd}(\sigma)} \qquad (1 \le k \le n). \tag{8b}$$

It is easy to see that $A_n^{\mathrm{cyc}}(x, y, t(x+y) \,|\, \alpha)$ is symmetric in x and y because the involution $\vartheta : \sigma \mapsto \sigma^{-1}$ for $\sigma \in \mathfrak{S}_n$ satisfies $(\mathrm{exc}, \mathrm{drop}, \mathrm{fix})\,\sigma = (\mathrm{drop}, \mathrm{exc}, \mathrm{fix})\,\sigma^{-1}$. We have the following (α, t)-analog of Chung-Graham-Knuth's symmetric Eulerian identity.

Theorem 3

For integers $a, b \ge 0$, we have

$$\left\langle {n \atop k} \right\rangle_{\alpha,t} := \left\langle {n \atop k} \right\rangle^{\mathrm{exc}}_{\alpha,t} = \left\langle {n \atop k} \right\rangle^{\mathrm{asc}}_{\alpha,t}, \tag{9a}$$

and

$$\sum_{k\ge 0}(\alpha t)^{a+b-k}\binom{a+b}{k}\left\langle {k \atop a} \right\rangle_{\alpha,t} = \sum_{k\ge 0}(\alpha t)^{a+b-k}\binom{a+b}{k}\left\langle {k \atop b} \right\rangle_{\alpha,t}, \tag{9b}$$

where $\left\langle {0 \atop k} \right\rangle_{\alpha,t} = \left\langle {k \atop 0} \right\rangle_{\alpha,t} = \delta_{k,0}$.

γ-positivity of (α, t)-Eulerian polynomials

Define the (α, t)-Eulerian polynomials $A_n(x, y, t \,|\, \alpha)$ and the (α, t)-binomial-Eulerian polynomials $\widetilde{A}_n(x, y, t \,|\, \alpha)$ by, respectively,

$$A_n(x, y, t \,|\, \alpha) := \sum_{\sigma\in\mathfrak{S}_{n+1}} x^{\mathrm{asc}(\sigma)}y^{\mathrm{des}(\sigma)}t^{\mathrm{lmaxda}(\sigma)+\mathrm{rmaxdd}(\sigma)}\alpha^{\mathrm{lmax}(\sigma)+\mathrm{rmax}(\sigma)-2}, \tag{10}$$

$$\widetilde{A}_n(x, y, t \,|\, \alpha) = \sum_{\sigma\in\mathcal{M}_{n+1}} x^{\mathrm{asc}(\sigma)}y^{\mathrm{des}(\sigma)}t^{\mathrm{lmaxda}(\sigma)+\mathrm{rmaxdd}(\sigma)}\alpha^{\mathrm{lmax}(\sigma)+\mathrm{rmax}(\sigma)-2}. \tag{11}$$

From Theorem 1 and 2, we derive the following combinatorial interpretations of the coefficients in the γ-expansion of $A_n(x, y, t \,|\, \alpha)$ and $\widetilde{A}_n(x, y, t \,|\, \alpha)$.

Theorem 4

For $0 \le j \le \lfloor n/2 \rfloor$, we have

$$A_n(x, y, t \,|\, \alpha) = \sum_{j=0}^{\lfloor n/2\rfloor}\gamma_{n,j}(\alpha, t)(xy)^j(x+y)^{n-2j}, \tag{12a}$$

$$\widetilde{A}_n(x, y, t \,|\, \alpha) = \sum_{j=0}^{\lfloor n/2\rfloor}\widetilde{\gamma}_{n,j}(\alpha, t)(xy)^j(x+y)^{n-2j}, \tag{12b}$$

where

$$\gamma_{n,j}(\alpha, t) = \sum_{\sigma\in\mathfrak{S}^{\mathrm{da}=0}_{n+1,\mathrm{asc}=j}} \alpha^{\mathrm{lmax}(\sigma)+\mathrm{rmax}(\sigma)-2}t^{\mathrm{rmaxdd}(\sigma)} \tag{13a}$$

$$= \sum_{\sigma\in\mathfrak{S}^{\mathrm{cda}=0}_{n,\mathrm{exc}=j}} 2^{\mathrm{cyc}(\sigma)-\mathrm{fix}(\sigma)}\alpha^{\mathrm{cyc}(\sigma)}t^{\mathrm{fix}(\sigma)}; \tag{13b}$$

and

$$\widetilde{\gamma}_{n,j}(\alpha, t) = \sum_{\sigma\in\mathcal{M}^{\mathrm{da}=0}_{n+1,\mathrm{asc}=j}} \alpha^{\mathrm{rmax}(\sigma)-1}t^{\mathrm{rmaxdd}(\sigma)} = \sum_{\sigma\in\mathfrak{S}^{\mathrm{cda}=0}_{n,\mathrm{exc}=j}} \alpha^{\mathrm{cyc}\,(\sigma)}t^{\mathrm{fix}(\sigma)}. \tag{13c}$$

with $\mathcal{X}^{\mathrm{st1}=0}_{n,\mathrm{st2}=j} := \{\sigma \in \mathfrak{S}_n : \mathrm{st1}(\sigma) = 0 \text{ and } \mathrm{st2}(\sigma) = j\}$ for $\mathcal{X} \in \{\mathfrak{S}, \mathcal{M}\}$.

γ-vetcor of (α, t)-Eulerian polynomials and cycle André permutations

For $0 \le j \le \lfloor n/2 \rfloor$, let $d_{n,j}(\alpha, t) = \gamma_{n,j}(\alpha, t)/2^j$, then, Eq. (12a) reads

$$A_n(x, y, t \,|\, \alpha) = \sum_{j=0}^{\lfloor n/2\rfloor}2^j d_{n,j}(\alpha, t)(xy)^j(x+y)^{n-2j}. \tag{14}$$

For a fixed $x \in [n]$, say that $\sigma \in \mathfrak{S}_n$ is an **André permutation of the first kind (resp. second kind)** if σ has no double descents, i.e., $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$, and each factorisation $u\,\lambda(x)\,x\,\rho(x)\,v$ of σ has property

- $\lambda(x) = \emptyset$ if $\rho(x) = \emptyset$,
- $\max(\lambda(x)) < \max(\rho(x))$ (resp. $\min(\rho(x)) < \min(\lambda(x))$) if $\lambda(x) \neq \emptyset$,

where $\lambda(x)$ and $\rho(x)$ are the the maximal contiguous subword immediately to the left (resp. right) of x whose letters are all greater than x. Let $\mathcal{A}_n^1$ (resp. $\mathcal{A}_n^2$) be the set of André permutations of the first (resp. second) kind in $\mathfrak{S}_n$. A right-to-left minimum (**rmin**) of σ is an element σ_i such that $\sigma_j > \sigma_i$ if $j > i$. A letter $\sigma_i \in [n]$ is a right-to-left-minimum-da (**rminda**) of σ if it is a double ascent and σ_i is a rmin. Let $C = (a_1, \dots, a_k)$ be a cycle of A with $a_1 = \min\{a_1, \dots, a_k\}$. Then, cycle C is called an ***André cycle*** if the word $a_2 \dots a_k$ is an André permutation of the first kind. We say that a permutation is a **cycle André permutation** if it is a product of disjoint André cycles. Let $\mathcal{CA}_n$ be the set of cycle André permutations of $[n]$.

Theorem 5

For $0 \le j \le \lfloor n/2 \rfloor$, we have

$$d_{n,j}(\alpha, t) = \sum_{\substack{\sigma\in\mathcal{CA}_n\\ \mathrm{drop}(\sigma)=j}} t^{\mathrm{fix}(\sigma)}\alpha^{\mathrm{cyc}(\sigma)} = \sum_{\substack{\sigma\in\mathcal{A}^{(i)}_{n+1}\\ \mathrm{des}(\sigma)=j}} t^{\mathrm{rminda}(\sigma)}\alpha^{\mathrm{rmin}(\sigma)-1}, \quad (i = 1, 2). \tag{15}$$

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