Poset Permutahedra

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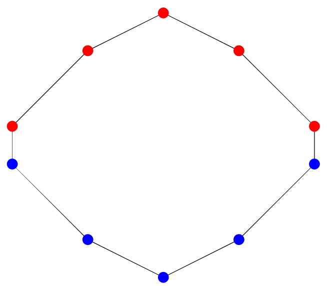
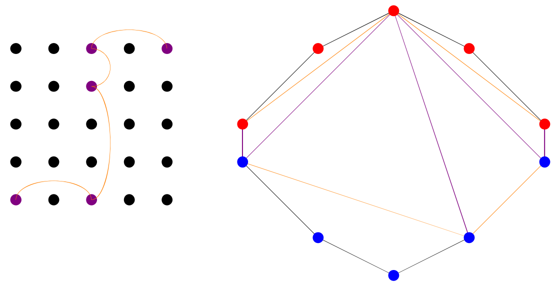
Color the vertices of a polygon red and blue:



One may consider a family of subdivisions of subpolygons of this polygon allowing edges between red and blue vertices:



Theorem

*Colorful triangulations of subpolygons of a polygon with m red vertices and n blue vertices are in bijection with lattice walks of **variable step length** from $(0,0)$ to (m,n).*

Motivating Question: Is there an **Associahedron** for this family of subdivisions?

Theorem

*For each m and n, there is a **simple** polytope $\Pi_{m,n}$ with face lattice given by colorful subdivisions of a polygon with m red vertices and n blue vertices. These polytopes appear in the following contexts:*

- *A toric model for cohomology rings of certain semisimple Hessenberg varieties*
- *Newton polytopes of mixed resultants*
- *Fiber polytopes for projections of products of simplices*

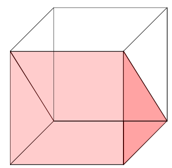
Question: How do we prove this?

Answer: We develop a new **general purpose construction** that we call **Poset Permutahedra**.

Given a poset $\mathcal{P} = ([n], \preceq)$, Stanley associated to it the Order Polytope:

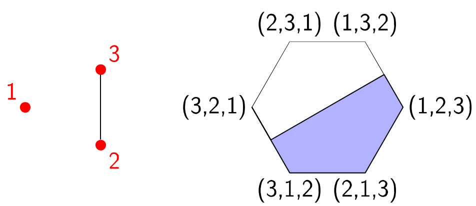
$$\mathcal{O}_{\mathcal{P}} = \{\mathbf{x} \in [0,1]^n : x_i \le x_j \text{ for all } i \preceq j\}.$$

For the poset $\mathcal{P}$ on $\{1,2,3\}$ with $2 \preceq 3$:



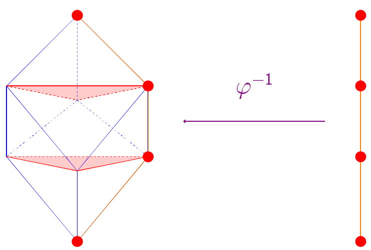
We define the **Poset Permutahedron** analogously:

$$\Pi_{\mathcal{P}} = \{\mathbf{x} \in \Pi_n : x_i \le x_j \text{ for all } i \preceq j\}.$$



Relating the two: Given a vector $\mathbf{c}$ and a polytope P, the **monotone path polytope** is

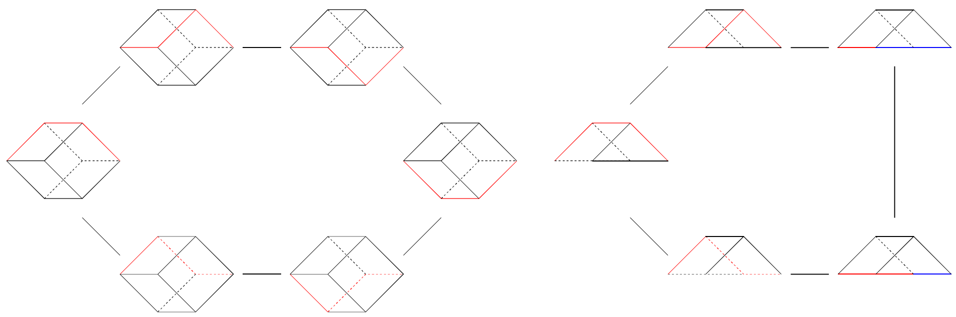
$$\Sigma_{\mathbf{c}}(P) = \sum_{v \in V(P)} P \cap \{\mathbf{x} : \mathbf{c}^{\intercal}\mathbf{x} = \mathbf{c}^{\intercal}\mathbf{v}\}.$$



Theorem (Billera and Sturmfels, 1992)

For $\mathbf{c} = (1,1,\ldots,1) \in \mathbb{R}^n$,

$$\Sigma_{\mathbf{c}}([0,1]^n) = \Pi_n.$$



Theorem

For $\mathbf{c} = (1,1,\ldots,1) \in \mathbb{R}^n$ and a poset $\mathcal{P} = ([n], \preceq)$,

$$\Sigma_{\mathbf{c}}(\mathcal{O}_{\mathcal{P}}) = \Pi_{\mathcal{P}}.$$

Combinatorial Consequences: We completely desceibe face lattices of these polytopes:

- Vertices of $\Pi_{\mathcal{P}}$ are **connected chains of order filters**
- Facets of $\Pi_{\mathcal{P}}$ correspond to cover relations and filters
- $\Pi_{\mathcal{P}}$ is simple if and only if the Hasse diagram of the poset $\mathcal{P}$ is a **forest**

Theorem

We have $\Pi_{m,n}$ is the poset permutahedron $\Pi_{\mathcal{P}}$, where $\mathcal{P}$ is the disjoint union of two chains of length m and n.