Double Boxes and Double Dimers

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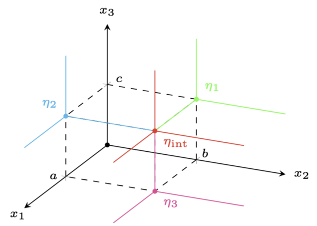
Abstract

In [1], Gholampour, Kool, and Young conjecture that the generating function for certain plane partition-like objects, called double-box configurations, is equal to a product of MacMahon's generating function for (boxed) plane partitions. In [2], Gholampour and Kool prove this result using geometric methods. We offer a combinatorial proof of this geometrically motivated result using the double-dimer model. We first give a correspondence between double-box configurations and double-dimer configurations on the hexagon lattice with a particular tripartite node pairing. Using this correspondence, we can apply graphical condensation and double-dimer condensation in our proof.

1 Double-box configurations

Fix $a, b, c \in \mathbb{N}$, and identify the point $(i, j, k) \in \mathbb{Z}^3$ with the unit cube (also called box) $[i, i+1] \times [j, j+1] \times [k, k+1]$. Let $\eta = (\eta_1, \eta_2, \eta_3)$ be a triple of plane partitions such that η_1 is based at the point $(0, b, c)$, η_2 is based at $(a, 0, c)$, and η_3 is based at $(a, b, 0)$ in $\mathbb{R}^3$.

Definition 1. We say that a box (i, j, k) is in the **intersection space** if $i \geq a, j \geq b$, and $k \geq c$. We denote boxes in the intersection space by η_{int}.



Definition 2. We say that a box $(i, j, k) \in \eta = (\eta_1, \eta_2, \eta_3)$ is:

- **type I** if $(i, j, k) \in \eta_m$ and $(i, j, k) \notin \eta_n, \eta_l$ for $\{m, n, l\} = \{1, 2, 3\}$.
- **type II** if $(i, j, k) \in \eta_m, \eta_n$ and $(i, j, k) \notin \eta_l$ for $\{m, n, l\} = \{1, 2, 3\}$.
- **type III** if $(i, j, k) \in \eta_1, \eta_2, \eta_3$.

Figure 1. Basepoints of plane partitions η_1, η_2, η_3 in $\mathbb{R}^3$.

For the following definitions, consider triples of plane partitions (η_1, η_2, η_3) placed in $\mathbb{R}^3$ as above.

Definition 3. We say that a triple of plane partitions (η_1, η_2, η_3) satisfies the **Overlap Condition** if every box in the intersection space η_{int} is a type II or a type III box. Two triples of plane partitions are called **compatible** if they have the same multiset of boxes.

Definition 4 ([1]). Let (η_1, η_2, η_3) be a triple of plane partitions that satisfies the Overlap Condition. The **double-box configuration** associated to (η_1, η_2, η_3) is the multiset of boxes in any triple of plane partitions compatible with (η_1, η_2, η_3). Let $DB_{a,b,c}$ denote the set of all double-box configurations.

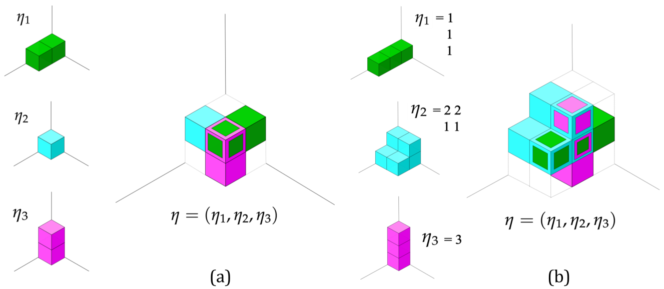


Figure 2. Examples of $\eta \in DB_{1,1,1}$. (a) One type II box at (1,1,1). (b) One type III box at (1,1,1), two type II boxes at (1,0,0) and (0,0,1).

Definition 5. The generating function for double-box configurations is given by

$$Z^{DB}_{a,b,c}(q) = \sum_{\eta \in DB_{a,b,c}} 2^m q^{|\eta|}$$

where 2^{m} is the number of compatible triples that yield $\eta \in DB_{a,b,c}$ for some $m \in \mathbb{N}$, and $|\eta| =$ #{type I boxes} + #{type II boxes} + 2#{type III boxes}.

Theorem 1. (Gholampour, Kool, Young [1])

$$Z^{DB}_{a,b,c}(q) = M(q)^2 M_{a,b,c}(q)$$

where $Z^{DB}_{a,b,c}(q)$ is the generating function for double-box configurations, and

$$M(q) = \prod_{i=1}^{\infty} \frac{1}{(1-q^i)^i}\ ,\ M_{a,b,c}(q) = \prod_{s=1}^{a} \prod_{t=1}^{b} \prod_{r=1}^{c} \frac{1-q^{s+t+r-1}}{1-q^{s+t+r-2}} \tag{1}$$

are MacMahon's generating functions for plane partitions and boxed $a \times b \times c$ plane partitions, respectively.

2 Tripartite double-dimer configurations

Definition 6. A **single-dimer configuration** on a graph $G = (V, E)$ is a collection of edges $E' \subseteq E$ such that every vertex in V is covered exactly once. Let $N \subset V$ be a set of **nodes**, that is, a special set of defined vertices (typically on the outer face of G). A **double-dimer configuration** on G with node set N is a multiset of E such that each vertex in $V \setminus N$ is covered exactly twice, and each node in N is covered exactly once.

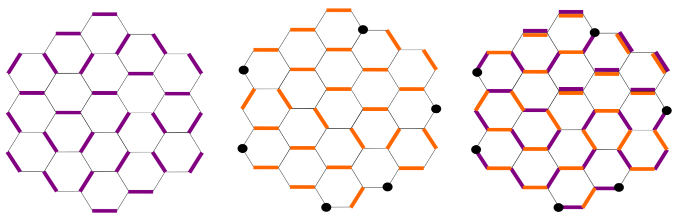
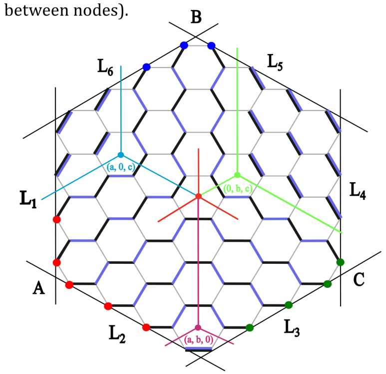


Figure 3. Left: single-dimer configuration on the hexagon graph. Middle: single-dimer configuration with nodes. Right: double-dimer configuration (loops, doubled edges, paths between nodes).



Definition 7. Let $a, b, c, n \in \mathbb{N}$. Let $H(n)$ be the hexagon graph of size $n \times n \times n$. Define a set of nodes N on the boundary of $H(n)$ by $N = R \cup G \cup B$, where:

$$\begin{aligned} R &= \{a \text{ nodes on } L_1 \text{ closest to } A\} \cup \\ &\quad\ \{c \text{ nodes on } L_2 \text{ closest to } A\} \\ G &= \{c \text{ nodes on } L_3 \text{ closest to } C\} \cup \\ &\quad\ \{b \text{ nodes on } L_4 \text{ closest to } C\} \\ B &= \{b \text{ nodes on } L_5 \text{ closest to } B\} \cup \\ &\quad\ \{a \text{ nodes on } L_6 \text{ closest to } B\} \end{aligned}$$

Color the R nodes red, the G nodes green, and the B nodes blue (see Figure 4).

Figure 4. A double-dimer configuration on $H(4)$ with $a = 2, b = 1$ and $c = 3$, and node set $N = R \cup G \cup B$.

Definition 8. Given $a, b, c \in \mathbb{N}$ and the node set N defined above, let $\sigma_{a,b,c}$ be the unique planar tripartite pairing of the nodes (that is, each node is paired with a node of a different color).

Definition 9. Let $DD(\sigma_{a,b,c})$ denote the set of all double-dimer configurations on the infinite hexagon graph such that for each $\pi \in DD(\sigma_{a,b,c})$, there exists $n \in \mathbb{N}$ such that π restricted to $H(n)$ has the node set N and the tripartite node pairing $\sigma_{a,b,c}$.

Definition 10. Define the generating function for elements in $DD(\sigma_{a,b,c})$ as

$$Z^{DD}_{a,b,c}(q) = \lim_{n \to \infty} \left(\sum_{\pi \in DD_n(\sigma_{a,b,c})} 2^{\ell(\pi)} w(\pi) \right)$$

where $\ell(\pi)$ is the number of closed loops of π on $H(n)$ and the configuration $\pi_0 \in DD_n(\sigma_{a,b,c})$ has minimal weight. The weight of a double-dimer configuration is the product of the edge weights of the chosen edges, where we choose the edge weights of the hexagon graph to reproduce the weighting in Definition 5.

3 Mapping double-box configurations to tripartite double-dimer configurations

There is a bijection between plane partitions and single-dimer configurations on the hexagon graph called the **folklore bijection**.

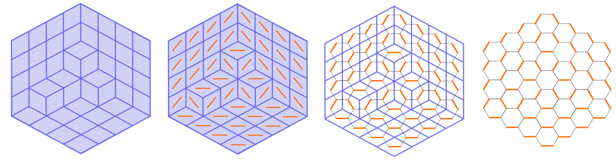


Figure 5. Folklore bijection between a plane partition η_i (leftmost) and single-dimer configurations on the hexagon graph D_{η_i} (rightmost).

Definition 11. Let $a, b, c \in \mathbb{N}$ and let $\eta = (\eta_1, \eta_2, \eta_3) \in DB_{a,b,c}$. Superimpose the single-dimer configurations corresponding to η_1, η_2 and η_3 (via the folklore bijection). Denote the triple-dimer configuration obtained in this way by by T_η (see Figure 6).

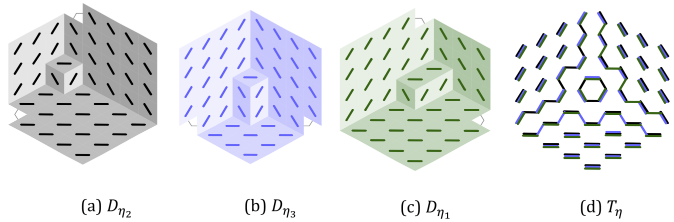


Figure 6. The tripartite triple-dimer configuration (rightmost) corresponding to the double-box configuration $\eta = (\eta_1, \eta_2, \eta_3)$ from Figure 2a.

Theorem 2. Let $a, b, c \in \mathbb{N}$ and let $\eta = (\eta_1, \eta_2, \eta_3) \in DB_{a,b,c}$. Removing the single-dimer configuration corresponding to the plane partition η_{int} based at (a, b, c) from T_η gives an element in $DD(\sigma_{a,b,c})$.

Theorem 3. $Z^{DB}_{a,b,c}(q) = Z^{DD}_{a,b,c}(q)$

Proof Sketch of Theorem 1: $Z^{DB}_{a,b,c}(q) = M(q)^2 M_{a,b,c}(q)$ (1)

We show that both sides of Equation 1 satisfy the recurrence relation

$$X(a,b,c)X(a+1,b+1,c) = X(a+1,b,c)X(a,b+1,c) + q^{a+b+1} X(a+1,b+1,c-1)X(a,b,c+1). \tag{2}$$

Using Theorem 3, we may replace the left-hand side of Equation 1, $Z^{DB}_{a,b,c}(q)$, with $Z^{DD}_{a,b,c}(q)$. Then we may apply a result of Jenne ([3]), called *double-dimer condensation*, to show that $Z^{DD}_{a,b,c}(q)$ satisfies Equation 2 . The right-hand side of Equation 1, $M(q)^2 M_{a,b,c}(q)$, satisfies the same recurrence by a result of Kuo ([6]), called *graphical condensation*. Finally, we show that both sides satisfy the same initial conditions.

References

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