On a super version of Thrall's problem

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Thrall's problem

- $\mathscr{L}(V) = \bigoplus_{n\geq 1}\mathscr{L}_n(V)$ is the free Lie algebra
- A tensor algebra decomposition:

$$T(V) \cong_{\mathrm{GL}(V)\text{-mod}} \bigoplus_{\lambda=(1^{a_1}2^{a_2}\cdots)} \mathscr{L}_\lambda(V),$$

$$\mathscr{L}_\lambda(V) = S^{a_1}(\mathscr{L}_1(V)) \otimes S^{a_2}(\mathscr{L}_2(V)) \otimes \cdots.$$

- Thrall (1942): What is the irreducible $\mathrm{GL}(V)$-decomposition of $\mathscr{L}_\lambda(V)$?

The classical case

Theorem (Lusztig—Stanley). The Frobenius character of the coinvariant algebra $R_n = \bigoplus_{k\geq 1} R_n^k$ is

$$\mathrm{grFrob}(R_n; q) = \sum_{T\in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} s_{\mathrm{sh}(T)}.$$

Theorem (Kraśkiewicz—Weyman 1985). The Schur—Weyl dual of $\mathscr{L}_n(V)$ is isomorphic to $\bigoplus_{k\geq 0} R_n^{1+kn}$. In particular, the multiplicity of V^μ in $\mathscr{L}_n(V)$ equals

$$|\{T \in \mathrm{SYT}(\mu) : \mathrm{maj}(T) \equiv_n 1\}|.$$

Example

$T =$

1	2	4
3	5	

$\in \mathrm{SYT}(3,2)$, $\mathrm{Des}(T) = \{2,4\}$,

$\mathrm{maj}(T) = 2 + 4 = 6.$

Super Thrall's problem

- $\widetilde{\mathscr{L}}(\mathbf{V}) = \bigoplus_{n,m\geq 0}\widetilde{\mathscr{L}}_{n,m}(\mathbf{V})$ is the **free Lie superalgebra** of $\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{V}_1$
- **(AS 2025)** A tensor superalgebra decomposition:

$$T(\mathbf{V}) \cong_{\mathrm{GL}(\mathbf{V})\text{-mod}} \bigoplus_{A=(a_{i,j})} \widetilde{\mathscr{L}}_A(\mathbf{V}),$$

$$\widetilde{\mathscr{L}}_A(\mathbf{V}) = \bigotimes_{i,j\geq 0} \Gamma_j^{a_{i,j}}(\widetilde{\mathscr{L}}_{i,j}(\mathbf{V})).$$

- Super Thrall's problem: What is the irreducible $\mathrm{GL}(\mathbf{V})$-decomposition of $\widetilde{\mathscr{L}}_A(\mathbf{V})$?

Super major index

Definition (AS 2025). Let $T \in \mathrm{SYT}(\mu)$ and $S \subseteq [n]$. The **super descent set** of (T, S) is

$$\mathrm{sDes}(T,S) = \{i : i \in \mathrm{Des}(T),\ i+1 \notin S\} \sqcup \{i : i \notin \mathrm{Des}(T),\ i \in S\}.$$

The **super major index** of (T, S) is

$$\mathrm{smaj}(T,S) = \sum_{i\in \mathrm{sDes}(T,S)} i.$$

Example

$T =$

1	3	4	6
2	5		
7			

$, S = \{2,3,7\}.$

$\mathrm{Des}(T) = \{1,4,6\}, \mathrm{sDes}(T,S) = \{4\} \sqcup \{2,3\},$

$\mathrm{smaj}(T,S) = 2 + 3 + 4 = 9.$

Results

Theorem (AS 2025). The principal specialization of the super Schur function $\widetilde{s}_\mu(\mathbf{x},\mathbf{y})$ is

$$\widetilde{s}_\mu(1, q, q^2, \ldots; t, tq, tq^2, \ldots) = \frac{\sum_{(T,S)\in \mathrm{SYT}_\pm(\mu)} q^{\mathrm{smaj}(T,S)} t^{|S|}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

Theorem (AS 2025). The multiplicity of V^μ in $\widetilde{\mathscr{L}}_{n,m}(V)$ is

$$\Big|\{(T,S) \in \mathrm{SYT}_\pm(\mu) : |S| = m,\ \mathrm{smaj}(T,S) \equiv_{n+m} 1\}\Big|.$$

Theorem (AS 2025). Let $\widetilde{R}_n = R_n \otimes \bigwedge\{\theta_1, \ldots, \theta_n\}$. Then

$$\mathrm{grFrob}(\widetilde{R}_n; q, t) = \sum_{(T,S)\in \mathrm{SYT}_\pm(n)} q^{\mathrm{smaj}(T,S)} t^{|S|} s_{\mathrm{sh}(T)},$$

so in particular, the Schur—Weyl dual of $\widetilde{\mathscr{L}}_{n,m}(V)$ is isomorphic to $\bigoplus_{k\geq 0} \widetilde{R}_n^{1+kn,m}$.

Other results:

- Power-sum expansion of $\mathrm{ch}(\widetilde{\mathscr{L}}_{n,m})$ (cf. Brandt 1944, Petrogradsky 2000)
- Schur—Weyl dual of $\widetilde{\mathscr{L}}_{n,m}(V)$ identified as an induced S_{n+m}-representation from C_{n+m} (cf. Klyachko 1974)