Orbits in the affine flag variety of type A

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Classical background: orbits in the flag variety

- Let G be a connected reductive algebraic group over $\mathbb{C}$. Let $B \subset G$ be a Borel subgroup of G.
- The well known *Bruhat decomposition* is $G = \bigsqcup_{w\in W} BwB$, where W is the *Weyl group* of G.
- The coset space G/B is also called the *flag variety*. So Bruhat decomposition is also about B-orbits in the flag variety.
- Now let $G = \mathsf{GL}(n,\mathbb{C})$, and B be the Borel subgroup consists of upper triangular matrices in G. In this case $W \simeq S_n$, the group of permutation of n elements.
- Let $K = \mathsf{O}(n,\mathbb{C})$. Then $G = \bigsqcup_{w\in I} KwB$, where I consists of involutions in S_n.
- Let $K = \mathsf{Sp}(n,\mathbb{C})$ (n even). Then $G = \bigsqcup_{w\in I^{\mathrm{fpf}}} KwB$, where I^{fpf} consists of fixed-point-free involutions in S_n.
- Let $K = \mathsf{GL}(p,\mathbb{C}) \times \mathsf{GL}(q,\mathbb{C})$ with $p+q=n$. Then $G = \bigsqcup_{w\in\mathcal{C}(U(p,q))} KwB$, where $\mathcal{C}(U(p,q))$ is the set of *(p,q)-clans*.
- The above K's satisfy $K = G^\theta$, where θ is a holomorphic involution. The three cases above correspond to $\theta(g) = (g^T)^{-1}$, $\theta(g) = (-Jg^TJ)^{-1}$ with n is even and $J = \begin{pmatrix} 0 & 1_{n/2} \\ -1_{n/2} & 0 \end{pmatrix}$, and $\theta(g) = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} g \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$ respectively. Here denote 1_n to be the n-by-n identity matrix.
- Study of (closure of) K-orbits is related to representation of real forms of G, Schubert calculus and equivariant cohomologies of the flag variety.

Orthogonal orbits

- Suppose $K = \mathsf{O}(n,\mathbb{C}((t))) = \{g \in G : g^Tg = 1_n\}$.
- Define an *affine permutation matrix* to be an n-by-n monomial matrix with integral powers of t as non-zero entries.
- Define *SymAPM_n* to be the set of all symmetric n-by-n affine permutation matrices. Define *$\mathsf{eSymAPM}_n$* to be the set of elements in SymAPM_n for which the sum of the powers of t is even.

Theorem

In the case where $K = \mathsf{O}(n,\mathbb{K}((t)))$ and $G = \mathsf{GL}(n,\mathbb{K}((t)))$, for each double coset $\mathcal{O} \in K\backslash G/B$, there exists a unique $w \in \mathsf{eSymAPM}_n$ such that $g^Tg = w$ for some $g \in \mathcal{O}$. Moreover, for each $w \in \mathsf{eSymAPM}_n$, the set of matrices g satisfying $g^Tg = w$ is non-empty and its elements lie in the same double coset.

- For each $w \in \mathsf{eSymAPM}_n$, there is an explicit formula for a matrix $g_w \in G$ such that $g_w^Tg_w = w$.

Corollary

The map $w \mapsto Kg_wB$ is a bijection between $\mathsf{eSymAPM}_n$ and $K\backslash G/B$.

- Define * to be the automorphism on affine permutation matrices by substituting t^{-1} in the places with t. Then every $w \in \mathsf{SymAPM}_n$ satisfies $w^* = w^{-1}$. We call these matrices as *extended affine twisted involutions*.
- Similarly, the set $\mathsf{eSymAPM}_n$ consists of all matrices in SymAPM_n for which the sum of the powers of t is an even integer. Therefore we call these matrices as *even extended affine twisted involutions*.
- Example: Suppose $n=3$. Then matrices in $\mathsf{eSymAPM}_3$ are in one of the following forms:

$$w_1 = \begin{pmatrix} t^a & 0 & 0 \\ 0 & t^b & 0 \\ 0 & 0 & t^c \end{pmatrix}, w_2 = \begin{pmatrix} 0 & t^a & 0 \\ t^a & 0 & 0 \\ 0 & 0 & t^b \end{pmatrix}, \quad \begin{pmatrix} t^b & 0 & 0 \\ 0 & 0 & t^a \\ 0 & t^a & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & t^a \\ 0 & t^b & 0 \\ t^a & 0 & 0 \end{pmatrix}.$$

In all of the above forms, the exponents a, b, c are integers. The sum $a+b+c$ is even for the first form and the integer b is even in the remaining forms. For example if a, b are odd and c is even in w_1, then

$$g_{w_1} = \begin{pmatrix} t^{\frac{a-1}{2}} & -(t-1)^{\frac12}t^{\frac{b-1}{2}} & 0 \\ t^{\frac{a-1}{2}}(t-1)^{\frac12} & t^{\frac{b-1}{2}} & 0 \\ 0 & 0 & t^{\frac{c}{2}} \end{pmatrix} \quad \text{and} \quad g_{w_2} = \begin{pmatrix} i & -it^a/2 & 0 \\ 1 & t^a/2 & 0 \\ 0 & 0 & t^{\frac{b}{2}} \end{pmatrix}.$$

Complications in special orthogonal orbits

- Let $G = \mathsf{SL}(n,\mathbb{K}((t)))$ and $K = \mathsf{SO}(n,\mathbb{K}((t))) = \{g \in \mathsf{SL}(n,\mathbb{K}((t))) : g^Tg = 1_n\}$.
- There is a definition of *$i\mathsf{SymAPM}_n$* $\subset G$ to be a subset of symmetric monomial matrices with entries t^a or $\pm it^a$.
- For each $w \in i\mathsf{SymAPM}_n$, we define explicitly $g_w \in \mathsf{SL}(n,\mathbb{K}((t)))$ satisfying $g_w^Tg_w = w$. Similar correspondence as above holds:

Corollary

The map $w \mapsto Kg_wB$ is a bijection between $i\mathsf{SymAPM}_n$ and $K\backslash G/B$.

- The matrices in $i\mathsf{SymAPM}_n$ can be indexed by *affine twisted involutions*, which are symmetric affine permutation matrices with sum of powers of t equal to 0.
- Example: Suppose $n=4$. Then matrices in $i\mathsf{SymAPM}_4$ are in one of the following forms:

$$w_1 = \begin{pmatrix} t^a & 0 & 0 & 0 \\ 0 & t^b & 0 & 0 \\ 0 & 0 & t^c & 0 \\ 0 & 0 & 0 & t^d \end{pmatrix}, w_2 = \begin{pmatrix} 0 & it^a & 0 & 0 \\ it^a & 0 & 0 & 0 \\ 0 & 0 & 0 & it^b \\ 0 & 0 & it^b & 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 & \text{-}it^a & 0 & 0 \\ \text{-}it^a & 0 & 0 & 0 \\ 0 & 0 & 0 & it^b \\ 0 & 0 & it^b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \pm it^a & 0 \\ 0 & 0 & 0 & it^b \\ \pm it^a & 0 & 0 & 0 \\ 0 & it^b & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & \pm it^a \\ 0 & 0 & it^b & 0 \\ 0 & it^b & 0 & 0 \\ \pm it^a & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & it^a & 0 & 0 \\ it^a & 0 & 0 & 0 \\ 0 & 0 & t^b & 0 \\ 0 & 0 & 0 & t^c \end{pmatrix}, \begin{pmatrix} 0 & 0 & it^a & 0 \\ 0 & t^b & 0 & 0 \\ it^a & 0 & 0 & 0 \\ 0 & 0 & 0 & t^c \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & it^a \\ 0 & t^b & 0 & 0 \\ 0 & 0 & t^c & 0 \\ it^a & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} t^b & 0 & 0 & 0 \\ 0 & 0 & it^a & 0 \\ 0 & it^a & 0 & 0 \\ 0 & 0 & 0 & t^c \end{pmatrix}, \begin{pmatrix} t^b & 0 & 0 & 0 \\ 0 & 0 & 0 & it^a \\ 0 & 0 & t^c & 0 \\ 0 & it^a & 0 & 0 \end{pmatrix}, \begin{pmatrix} t^b & 0 & 0 & 0 \\ 0 & t^c & 0 & 0 \\ 0 & 0 & 0 & it^a \\ 0 & 0 & it^a & 0 \end{pmatrix}.$$

In all of the above forms, the exponents in t's are integers and add up to zero. Suppose a, b are odd, and c, d are even in w_1. Then

$$g_{w_1} = \begin{pmatrix} t^{\frac{a-1}{2}} & -(t-1)^{\frac12}t^{\frac{b-1}{2}} & 0 & 0 \\ t^{\frac{a-1}{2}}(t-1)^{\frac12} & t^{\frac{b-1}{2}} & 0 & 0 \\ 0 & 0 & t^{\frac{c}{2}} & 0 \\ 0 & 0 & 0 & t^{\frac{d}{2}} \end{pmatrix} \quad \text{and}$$

$$g_{w_2} = \begin{pmatrix} t^a/2 & i & 0 & 0 \\ it^a/2 & 1 & 0 & 0 \\ 0 & 0 & t^b/2 & i \\ 0 & 0 & it^b/2 & 1 \end{pmatrix} \quad \text{and} \quad g_{w_3} = \begin{pmatrix} i & -t^a/2 & 0 & 0 \\ 1 & -it^a/2 & 0 & 0 \\ 0 & 0 & t^b/2 & i \\ 0 & 0 & it^b/2 & 1 \end{pmatrix}.$$

Affine analogs: orbits in the affine flag variety

- Let $\mathbb{K}$ be a quadratically closed field, i.e. a field of char. not equal to 2 in which every element has a square root.
- Let $\mathbb{K}((t))$ be the field of formal Laurent series in t consisting of all the formal sums $\sum_{i\geq N}^{\infty} a_it^i$, in which $N \in \mathbb{Z}$ and $a_i \in \mathbb{K}$ for $i \geq N$.
- Let $\mathbb{K}[[t]]$ be the ring of formal power series consisting of all the formal sums $\sum_{i\geq 0}^{\infty} a_it^i$, in which $a_i \in \mathbb{K}$.
- Redefine $G = \mathsf{GL}(n,\mathbb{K}((t)))$ to be the group of invertible n-by-n matrices over $\mathbb{K}((t))$.
- Redefine B to be the subgroup consisting of all upper triangular modulo t matrices in $\mathsf{GL}(n,\mathbb{K}[[t]])$, that is, invertible matrices with entries in $\mathbb{K}[[t]]$ that become upper triangular if we set $t=0$ for these matrices.
- The G above is the *(algebraic) loop group* of $\mathsf{GL}(n,\mathbb{K})$ and B is an *Iwahori subgroup*.
- The *affine Bruhat decomposition* is written as
$$G = \bigsqcup_{w\in\widetilde{W}} BwB,$$
where $\widetilde{W}$ is the *affine Weyl group* of G, which is isomorphic to a semidirect product of the symmetric group S_n of permutations of n elements and $\mathbb{Z}^n$ of n-tuples of integers.
- The set of cosets G/B is often called the *affine flag variety*.
- In this work, we investigate the K-orbits in G/B, where $K = \mathsf{O}(n,\mathbb{K}((t)))$, $\mathsf{Sp}(n,\mathbb{K}((t)))$ or $\mathsf{GL}(p,\mathbb{K}((t))) \times \mathsf{GL}(q,\mathbb{K}((t)))$. We also consider the $\mathsf{SO}(n,\mathbb{K}((t)))$-orbits in the affine flag variety of $\mathsf{SL}(n,\mathbb{K}((t)))$.

Symplectic orbits

- Let $G = \mathsf{GL}(2n,\mathbb{K}((t)))$, and $K = \mathsf{Sp}(2n,\mathbb{K}((t))) = \{g \in \mathsf{GL}(2n,\mathbb{K}((t))) : g^TJg = J\}$.
- The set *$\mathsf{SkewAPM}_{2n}$* consists of all skew-symmetric $2n$-by-$2n$ monomial matrices whose non-zero entries above the diagonal are integral powers of t.

Theorem

In the case where $K = \mathsf{Sp}(2n,\mathbb{K}((t)))$ and $G = \mathsf{GL}(2n,\mathbb{K}((t)))$, for each double coset $\mathcal{O} \in K\backslash G/B$, there exists a unique $w \in \mathsf{SkewAPM}_{2n}$ such that $g^TJg = w$ for some $g \in \mathcal{O}$. Moreover, for each $w \in \mathsf{SkewAPM}_{2n}$, the set of matrices g satisfying $g^TJg = w$ is non-empty and its elements lie in the double coset.

- For each $w \in \mathsf{SkewAPM}_{2n}$, there is an explicit formula for a matrix $g_w \in \mathsf{GL}(2n,\mathbb{K}((t)))$ such that $g_w^TJg_w = w$.

Corollary

The map $w \mapsto Kg_wB$ is a bijection between $\mathsf{SkewAPM}_{2n}$ and $K\backslash G/B$.

- The matrices in $\mathsf{SkewAPM}_{2n}$ can be indexed by the set of *fixed-point-free extended affine twisted involutions*, consisting of symmetric affine permutation matrices with no non-zero diagonal entries.
- Example: Suppose $n=2$. Then matrices in $\mathsf{SkewAPM}_4$ are in one of the following forms:

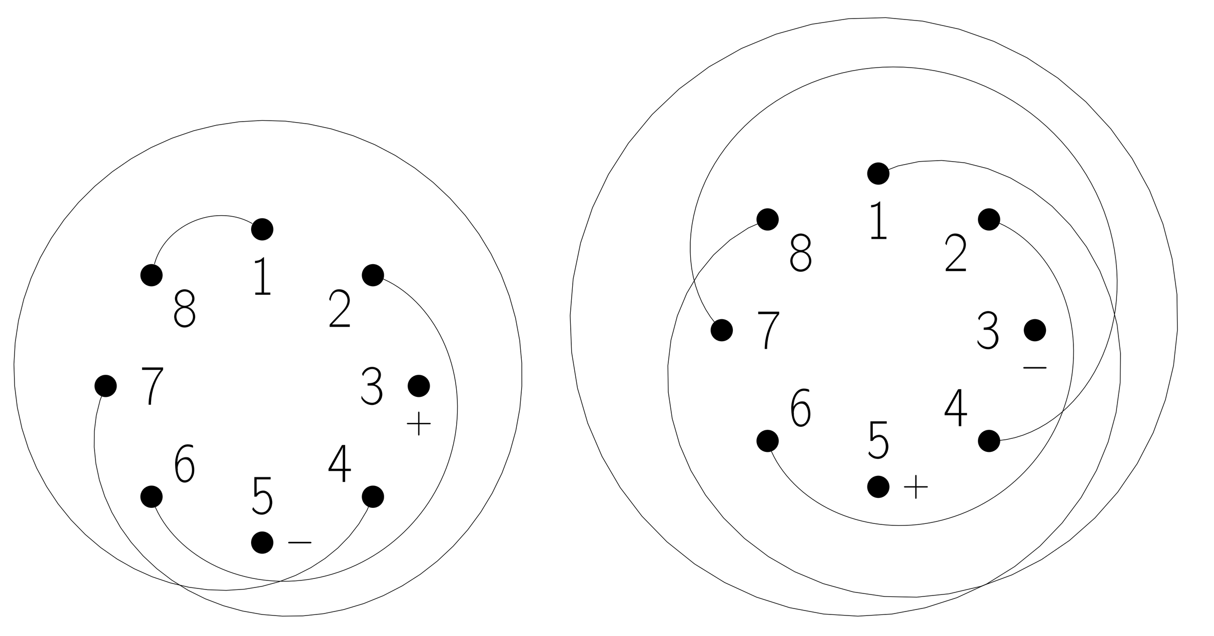
$$w_1 = \begin{pmatrix} 0 & t^a & 0 & 0 \\ -t^a & 0 & 0 & 0 \\ 0 & 0 & 0 & t^b \\ 0 & 0 & -t^b & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & t^a & 0 \\ 0 & 0 & 0 & t^b \\ -t^a & 0 & 0 & 0 \\ 0 & -t^b & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & t^a \\ 0 & 0 & t^b & 0 \\ 0 & -t^b & 0 & 0 \\ -t^a & 0 & 0 & 0 \end{pmatrix}.$$

Here a and b are integers. It holds that

$$g_{w_1} = \begin{pmatrix} t^a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t^b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Product group orbits

- Suppose $K = \mathsf{GL}(p,\mathbb{K}((t))) \times \mathsf{GL}(q,\mathbb{K}((t))) = \left\{\left(\begin{smallmatrix} k_1 & 0 \\ 0 & k_2 \end{smallmatrix}\right) : k_1 \in \mathsf{GL}(p,\mathbb{K}((t))), k_2 \in \mathsf{GL}(q,\mathbb{K}((t)))\right\}$ with $p+q=n$.
- An *affine (p,q)-clan* is a $\mathbb{Z}$-indexed sequence $c = (\dots, c_1, c_2, c_3, \dots)$ with $n = p+q$ encoding an *affine involution* with $+$ or $-$ signs assigned to the fixed points, s.t. $\#\{i \in [n] : c_i = +\} - \#\{i \in [n] : c_i = -\} = p-q$.
- For example, the affine $(1,1)$-clans are $(+,-)$, $(-,+)$ and $(1, 1+2k)$ for $k \in \mathbb{Z}$. The affine $(2,1)$-clans are $(+,+,-)$, $(+,-,+)$, $(-,+,+)$, $(1, 1+3k, +)$, $(+, 1, 1+3k)$ and $(1, +, 1+3k)$ for $k \in \mathbb{Z}$.
- Below are *winding diagrams* for affine $(4,4)$-clans with $(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8) = (1, 2, +, 3, -, 2, 2-8, 1+8)$ and $(1, 2, -, 3, +, 2, 2+8, 1-8)$ respectively.



- For every affine (p,q)-clan $c = (\dots, c_1, c_2, \dots, c_n, \dots)$, there is an inductive procedure defining an *affine (p,q)-clan matrix* in $\mathsf{GL}(n,\mathbb{K}((t)))$.

Theorem

Suppose $G = \mathsf{GL}(n,\mathbb{K}((t)))$, B the Iwahori subgroup of G, and $K = \mathsf{GL}(p,\mathbb{K}((t))) \times \mathsf{GL}(q,\mathbb{K}((t)))$. The affine (p,q)-clan matrices are distinct double coset representatives of the double cosets in $K\backslash G/B$.

- Example: Suppose $n=3$, $p=2$ and $q=1$, and $a \in \mathbb{Z}_{\leq 0}$, $b \in \mathbb{Z}_{<0}$. Then the following affine $(2,1)$-clan matrices are distinct double coset representatives in $K\backslash G/B$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t^a & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & t^b \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ t^a & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & t^b & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t^a & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & t^b \end{pmatrix}.$$

The affine $(2,1)$-clan matrices above correspond to the affine $(2,1)$-clans $(-,+,+)$, $(+,-,+)$ and $(+,+,-)$, $(1, 1+3a, +)$, $(1, 1-3b, +)$, $(+, 1, 1+3a)$, $(+, 1, 1-3b)$, $(1, +, 1+3a)$ and $(1, +, 1-3b)$ respectively.

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