GBG-rank Generating Functions for Integer Partitions

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Introduction of Integer Partitions

- Integer partition: $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_\ell) \in \mathbb{N}^\ell$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$.
- Young diagram: A left-justified array of boxes where the i-th row has λ_i boxes, corresponding to the parts of a partition.
- Hook length: The total number of boxes to the right(arm), below(leg), and the box itself, based on the box $(i, j) \in \lambda$.
- Frobenius symbol $\mathfrak{F}(\lambda)$: $(a_1+1, \ldots, a_s+1 \mid b_1, \ldots, b_s)$ where a_i and b_i are the arm and leg lengths of the box $(i, i) \in \lambda$.
- t-residue diagram: The Young diagram where each box (i, j) is filled with $j - i \bmod t$.
- GBG-rank modulo t: $\omega_t(\lambda) = \sum_{(i,j)\in\lambda} \zeta^{j-i}$ where $\zeta = \zeta_t = e^{2\pi i/t}$.

8				
	5			
		1		

Figure 1. The Young diagram Y_λ of the partition $\lambda = (5, 4, 3, 2)$.
The numbers on the diagonal show their hook lengths 8, 5, and 1. Yellow and green indicate arm and leg parts, respectively.

0	1	2	0	1
2	0	1	2	
1	2	0		
0	1			

Figure 2. The 3-residue diagram Y_λ of the partition $\lambda = (5, 4, 3, 2)$.
Each box is labeled with its residue value $\bmod 3$.
The GBG-rank $\omega_3(\lambda)$ is given by the sum $\sum_{(i,j)\in\lambda} \zeta_3^{j-i} = 1 + \zeta_3$.

Theorem: GBG-rank Generating functions

- (Infinite Case) Let $t \geq 2$ be a fixed prime, and let ζ be a primitive t-th root of unity. For $\omega = k\zeta^\ell$, define

$$G_t(\omega, q) := \sum_{\substack{\lambda\in\mathcal{P} \\ \omega_t(\lambda)=\omega}} q^{|\lambda|}.$$

For $k \in \mathbb{Z}$ and $\ell \in \{0, 1, \cdots, t-1\}$,

$$G_t(k\zeta^\ell, q) = \begin{cases} \dfrac{q^{tk^2-(t-1)k}}{(q^t;q^t)_\infty^t} & \text{if } \ell = 0, \\ \dfrac{q^{tk^2+k}}{(q^t;q^t)_\infty^t} & \text{if } 1 \leq \ell \leq t-1, \end{cases}$$

where $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$, $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$.

- (Finite Case) Define

$$G_{M,N,t}(\omega, q) := \sum_{\substack{\lambda\in\mathcal{P} \\ \lambda_1\leq N,\ \ell(\lambda)\leq M \\ \omega_t(\lambda)=\omega}} q^{|\lambda|}.$$

For $k \in \mathbb{Z}$, $M, N \in \mathbb{N}$ and $\ell \in \{0, \cdots, t-1\}$,

$$G_{M,N,t}(k\zeta^\ell, q) =$$
$$\begin{cases} q^{tk^2-(t-1)k} \begin{bmatrix} N_{t,1}+M_{t,1} \\ N_{t,1}-k \end{bmatrix}_{q^t} \begin{bmatrix} N_{t,t}+M_{t,t} \\ N_{t,t}+k \end{bmatrix}_{q^t} \prod_{j=2}^{t-1} \begin{bmatrix} N_{t,j}+M_{t,j} \\ N_{t,j} \end{bmatrix}_{q^t} & \ell = 0, \\ q^{tk^2+k} \begin{bmatrix} N_{t,\ell+1}+M_{t,\ell+1} \\ N_{t,\ell+1}-k \end{bmatrix}_{q^t} \begin{bmatrix} N_{t,\ell}+M_{t,\ell} \\ N_{t,\ell}+k \end{bmatrix}_{q^t} \prod_{\substack{1\leq j\leq t \\ j\neq \ell,\ell+1}} \begin{bmatrix} N_{t,j}+M_{t,j} \\ N_{t,j} \end{bmatrix}_{q^t} & 1 \leq \ell \leq t-1, \end{cases}$$

where $M_{t,j} := \left\lfloor \frac{M+j-1}{t} \right\rfloor$, $N_{t,j} := \left\lfloor \frac{N-j}{t} \right\rfloor + 1$, $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$.

References

[1] A. Berkovich and F. G. Garvan, The BG-rank of a Partition and Its Applications, *Advances in Applied Mathematics*, **40** (2008), no. 3, pp. 377–400.

[2] A. Berkovich and A. Dhar, On Partitions with Bounded Largest Part and Fixed Integral GBG-rank Modulo Primes, *Annals of Combinatorics*, to appear (2024), pp. 1–14.

Sketch of Proof

- The GBG-rank generating function is expressed using Frobenius symbols as

$$\sum_{\lambda\in\mathcal{P}} x^{\omega_t(\lambda)} q^{|\lambda|} = [z^0] \prod_{j=1}^{t} (-x^{r_j} zq; q)_\infty (-x^{-r_j} z^{-1} q^{t-j}; q)_\infty,$$

where $r_j := \sum_{i=0}^{j-1} \zeta^i$, $\zeta = \zeta_t$.

- Applying the Jacobi triple product identity to each factor, we obtain a multiple sum:

$$\sum_{\lambda\in\mathcal{P}} x^{\omega_t(\lambda)} q^{|\lambda|} = \frac{1}{(q^t;q^t)_\infty^t} \sum_{\substack{n_1+\cdots+n_t=0 \\ n_1,\cdots,n_t=-\infty}}^{\infty} x^{\sum_{j=1}^t n_j r_j} q^{\sum_{j=1}^t \left(\frac{tn_j(n_j-1)}{2}+jn_j\right)}.$$

- The exponent of x becomes an integer if and only if $n_2 = \cdots = n_{t-1} = 0$, and $n_t = -n_1$, leading to

$$G_t(k, q) = \frac{q^{tk^2-(t-1)k}}{(q^t;q^t)_\infty^t}.$$

- The same method applies to other values of ℓ.
- The finite cases can be obtained using the following finite form of Jacobi triple product identity: For $z \neq 0$ and $N, M \in \mathbb{N} \cup \{0\}$,

$$\sum_{n=-M}^{N} \begin{bmatrix} N+M \\ N-n \end{bmatrix}_{q^t} z^n q^{n(n+1)/2} = (-zq;q)_N (-1/z;q)_M.$$

Theorem: Self-conjugate and Doubled distinct

Let $GSC_t(\omega, q), GDD_t(\omega, q)$ be the GBG-rank generating functions modulo t for self-conjugate partitions and doubled distinct partitions respectively.
For an odd prime t and an integer k,

$$GSC_t(k, q) = \frac{(-q^t;q^{2t})_\infty q^{tk^2-(t-1)k}}{(q^{2t};q^{2t})_\infty^{(t-1)/2}},$$

$$GDD_t(k+k\zeta, q) = \frac{(-q^{2t};q^{2t})_\infty q^{tk^2-(t-2)k}}{(q^{2t};q^{2t})_\infty^{(t-1)/2}}.$$

1	ζ	ζ^2	ζ^3	ζ^4
ζ^{-1}	1	ζ		
ζ^{-2}	ζ^{-1}	1		
ζ^{-3}				
ζ^{-4}				

Figure 3. The GBG-rank diagram Y_λ of the self-conjugate partition $\lambda = (5, 3, 3, 3, 1)$.
ζ^n and ζ^{-n} always come in pairs off the main diagonal, so the GBG-rank of any self-conjugate partition is real.

1	ζ	ζ^2	ζ^3	ζ^4
ζ^{-1}	1	ζ	ζ^2	
ζ^{-2}	ζ^{-1}	1	ζ	
ζ^{-3}				

Figure 4. The GBG-rank diagram Y_λ of the doubled distinct partition $\lambda = (5, 4, 4, 1)$.
Similarly, ζ^n and ζ^{-n+1} always appear in pairs, so the GBG-rank value of doubled distinct partitions is of the form $(1 + \zeta) \cdot r$ for some $r \in \mathbb{R}$.

Future Work

- Investigate GBG-rank generating functions for other restricted partitions.
- Develop formulas for general (nonprime) values of t.

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