Interpreting the chromatic polynomial coefficients via hyperplane arrangements

NEHA GOREGAOKAR*

Background

Stanley '73: For a graph G,

$$\chi_G(-1) = \#\text{ of acyclic orientations of } G.$$

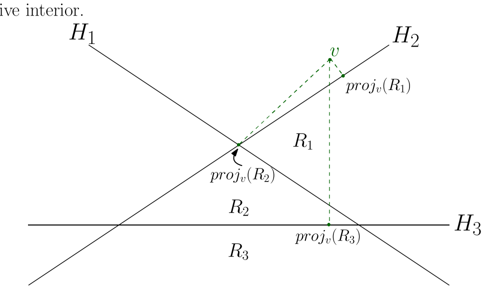
Zaslavsky '74: For a hyperplane arrangement $\mathcal{A}$,

$$(-1)^n \chi_{\mathcal{A}}(-1) = \#\text{ of regions of } \mathcal{A}.$$

Theorem: Let $G = ([n], E)$ be a graph and let $\mathcal{A}_G$ be the corresponding graphical arrangement. Then,

$$\chi_G(t) = \chi_{\mathcal{A}_G}(t).$$

Projection Statistic: $v \in \mathbb{R}^n$, R a region of $\mathcal{A}$,
$\text{proj}_v(R)$ = the unique point in R having minimum Euclidean distance from v.
$\text{pd}_v(R)$ = dimension of the unique face of R that contains $\text{proj}_v(R)$ in its relative interior.



Lofano, Paolini '21, Kabluchko '23:

$$(-1)^{n-k}[t^k]\chi_{\mathcal{A}}(t) = \#\text{ of regions } R \text{ of } \mathcal{A} \text{ with } \text{pd}_v(R) = k$$

Greene, Zaslavsky '83:

$$(-1)^{n-k}[t^k]\chi_G(t) = \#\text{ of acyclic orientations of } G \text{ with } k \text{ source components.}$$

Question

Given an arrangement $\mathcal{A}$ which has a combinatorial labeling of the regions, find $v \in \mathbb{R}^n$ such that $\text{pd}_v(R)$ is a "nice" combinatorial statistic on the object labeling the region R.

Main Results

- For the braid arrangement, the projection statistic coincides with the RLmin statistic on permutations.
- For graphical arrangements, the projection statistic coincides with the Greene and Zaslavsky statistic.
- The interpretation of the projection statistic for the braid arrangement generalizes to natural unit interval graphs.

*Dept. of Mathematics, Brandeis University, Waltham, USA.
ngoregaokar@brandeis.edu

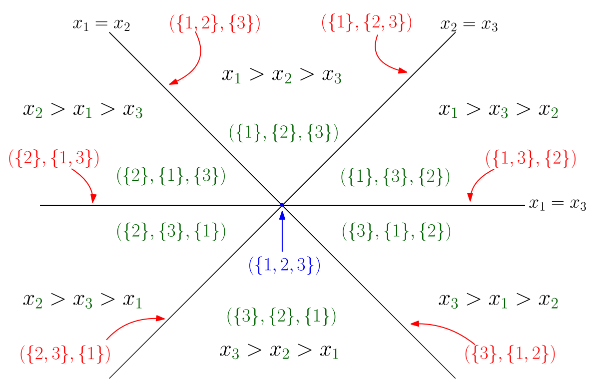
The Braid Arrangement

The Arrangement: The *braid arrangement* of dimension n is

$$\mathcal{B}_n = \bigcup_{1\le i<j\le n} \{x_i = x_j\}.$$

Labeling. For the braid arrangement $\mathcal{B}_n$,

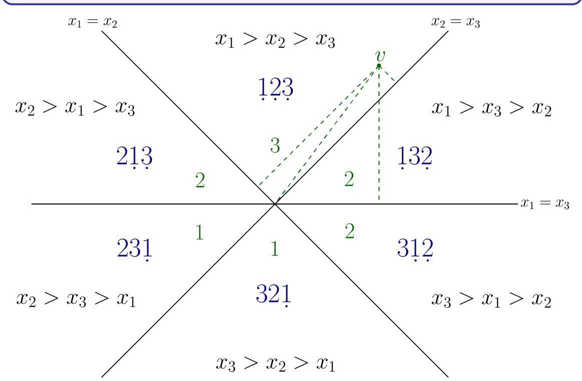
- Regions are labeled by permutations of $[n]$.
- Flats are labeled by partitions of $[n]$.
- Faces are labeled by ordered partitions of $[n]$.



Theorem.
For $v \in \mathbb{R}^n$ be such that $v_1 > \ldots > v_n$ and $v_i - v_{i+1} > n(v_{i+1} - v_n)$, and $\sigma \in \mathfrak{S}_n$,

$$\text{pd}_v(R_\sigma) = \#\text{ of right-to-left minima of } \sigma,$$

where R_σ is the region of $\mathcal{B}_n$ labeled by σ.



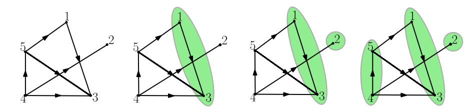
Lemma.
For $v \in \mathbb{R}^n$ as above and $\Pi = (B_1, \ldots, B_k)$ an ordered partition of $[n]$, $\text{proj}_v(\text{span}(F_\Pi))$ lies in the relative interior of F_Π if and only if $\min(B_i) < \min(B_{i+1})$ for all $i < k$.

Graphical Arrangements

The Arrangement: Let $G = ([n], E)$ be a graph. The *graphical arrangement* is

$$\mathcal{A}_G = \bigcup_{\{i,j\}\in E} \{x_i = x_j\}.$$

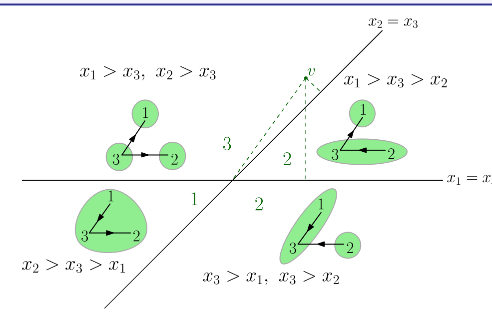
Labeling. Associate to each region R of $\mathcal{A}_G$ the acyclic orientation γ_R of G, given by directing the edge $\{i,j\}$ towards i if $x_i \ge x_j$ in R.
Source Components.
1st component S_1 = 1-reachable vertices
kth components S_k = r-reachable vertices (not in S_i for $i < k$) where $r = \min([n] \setminus \bigcup_{i<k} S_i)$.



Theorem.
For $v \in \mathbb{R}^n$ such that $v_i > (6n^2 + 1)v_{i+1}$ and $v_n > 0$, and R a region of the graphical arrangement $\mathcal{A}_G$,

$$\text{pd}_v(R) = \#\text{ of source components of } \gamma_R.$$

In fact, the face of R that $\text{proj}_v(R)$ lies in the interior of is the face labeled by the ordered partition of source components of γ_R.



Natural Unit Interval Graphs

Definition: A graph $G = ([n], E)$ is a *natural unit interval graph* if

$$\forall\, 1 \le i < k < j \le n, \text{ if } \{i,j\} \in E, \text{ then } \{i,k\},\{k,j\} \in E.$$

Theorem.
For $G = ([n], E)$ a natural unit interval graph, and $v \in \mathbb{R}^n$ as above,

$$\text{pd}_v(R) = \#\text{ of right-to-left minima of } \sigma.$$

where σ is the lexicographic minimum permutation associated to the region R of $\mathcal{A}_G$.

Corollary.
Let $G = ([n], E)$ be a natural unit interval graph. Then, $\chi_{\mathcal{A}_G}(q) = \prod_{j=1}^{n}(q - c_j)$, where $c_j = |\{i < j \mid \{i,j\} \in E\}|$ for all $j \in [n]$.