Grothendieck Shenanigans
Permutons from pipe dreams via integrable probability

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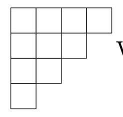
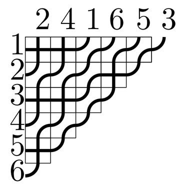
Abstract

We study random permutations corresponding to pipe dreams. Our main model is motivated by the Grothendieck polynomials with parameter $\beta = 1$ arising in the K-theory of the flag variety. permutation is proportional to the principal specialization of its Grothendieck polynomial. By mapping this random permutation to a version of TASEP, we describe the limiting permuton and fluctuations around it as the order n of the permutation grows to infinity. The fluctuations are of order $n^{\frac{1}{3}}$ and have the Tracy–Widom GUE distribution, which places this algebraic (K-theoretic) model into the KPZ universality class. Inspired by Stanley's question for the maximal value of principal specializations of Schubert polynomials, we resolve the analogous question for $\beta = 1$ Grothendieck polynomials, and provide bounds for general β. This analysis uses a correspondence with the free fermion six-vertex model, and the frozen boundary of the Aztec diamond.

Background: models

Tilings of a triangle  with ┼ and ╯╭

Schubert

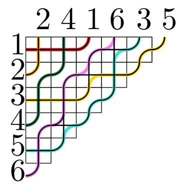


- No two pipes intersect more than once
- number of configurations $U(n)$

1, 2, 7, 41, 393, 6080, 150371, 5903710, 365973851, . . . ?

- What is the typical permutation?s
- What is the image of n? What is the average number of inversions?

Grothendieck



- Resolve after first intersection (Demazure product)
- number of configurations $b_n = 2^{\binom{n}{2}}$
- Typical permutation 

Background: Schubert polynomials

Schubert polynomial for a permutation $w \in S_n$: $\mathfrak{S}_w(x_1, \ldots, x_n)$

Origins: cohomology cycles of Schubert classes in flag varieties. (Lascoux–Schutzenberger 82)

Definition

$\partial_i f = \frac{f(\ldots,x_i,x_{i+1},\ldots)-f(\ldots,x_{i+1},x_i,\ldots)}{x_i - x_{i+1}}$

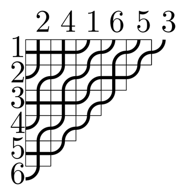
$w_0 := n(n-1)\ldots 21$

$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$

$\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i}$ when $w_i < w_{i+1}$

Pipe dreams (Bergeron-Billey 93):

$$\mathfrak{S}_w = \sum_{RPD(i \to w_i)} \prod_i x_i^{\#(i,j)=+}$$



$\to$ monomial $x_1^2 x_2 x_3^2 x_5$ in $\mathfrak{S}_w^\beta(x_1, \ldots, x_6)$ for $w = 241653$.

Background: Grothendieck polynomials

Grothendieck polynomial for a permutation $w \in S_n$: $\mathfrak{G}_w(x_1, \ldots, x_n)$

Origins: K-theory of the flag variety.

Definition

Demazure difference operators:

$\pi_i f := \frac{(1-\beta x_{i+1})f - (1-\beta x_i) f(\ldots,x_{i+1},x_i,\ldots)}{x_i - x_{i+1}}$,

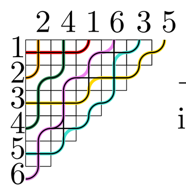
$\mathfrak{G}_w^\beta = \pi_{w^{-1}w_0}(x_1^{n-1}x_2^{n-2}\cdots x_{n-1})$

- $\mathfrak{S}_w(x_1, \ldots, x_n) = \mathfrak{G}_w^0(x_1, \ldots, x_n)$

Pipe dreams

$$\mathfrak{G}_w^\beta = \sum_{D \in PD(i \to w_i)} \beta^{-\ell(w)} \prod_i (\beta x_i)^{\#(i,j)=+}$$

Permutation w read by "resolving" double intersections:



$\to$ monomial $\beta^4 x_1^3 x_2^2 x_3^2 x_4 x_5$ in $\mathfrak{G}_w^\beta(x_1, \ldots, x_6)$ for $w = 241635$.

Motivation: Schubert Shenanigans (Stanley 2018)

$$u(n) := \max_{w \in S_n} \mathfrak{S}_w(1^n) \qquad U(n) := \sum_{w \in S_n} \mathfrak{S}_w(1^n) (= \# \text{ RC graphs})$$

Conjecture (Stanley 2018)

There exists a constant c, such that $c = \lim_{n\to\infty} \frac{1}{n^2} \log_2 u(n) = \lim_{n\to\infty} \frac{1}{n^2} \log_2 U(n)$.

Layered permutations L_n: $w(b_k, b_{k-1}, \ldots, b_1) := \big(w_0(b_k), b_k + w_0(b_{k-1}), \ldots, n - b_1 + w_0(b_1)\big)$

Conjecture (Merzon-Smirnov 2014)

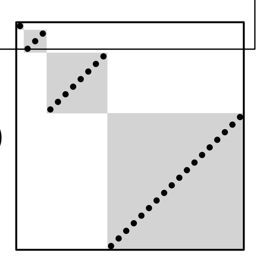
For every n, all permutations w attaining the maximum $u(n)$ are layered permutations. In particular, $u(n) = v(n)$ and $c = \gamma/\ln 2$.

Theorem (Morales-Pak-Panova 2018)

Let $v(n) := \max_{w \in L_n} \mathfrak{S}_w(1^n)$. Then there is a limit

$$\lim_{n\to\infty} \frac{1}{n^2} \log_2 v(n) = \frac{\gamma}{\log 2} \approx 0.2932362762,$$

$w(1,3,8,18)$



and the maximum value $v(n)$ is achieved at $w(\ldots, b_2, b_1)$, where $b_i \sim \alpha^{i-1}(1-\alpha)n$ as $n \to \infty$, for every fixed i, and where $\alpha \approx 0.4331818312$ is a universal constant.

- Best bounds for c: $0.2932 \le c \le 0.3774$. upper-bound $u(n) \le \#\mathrm{ASM}_n$ Weigandt 2019

Main result I Layered permutations achieve the limit.

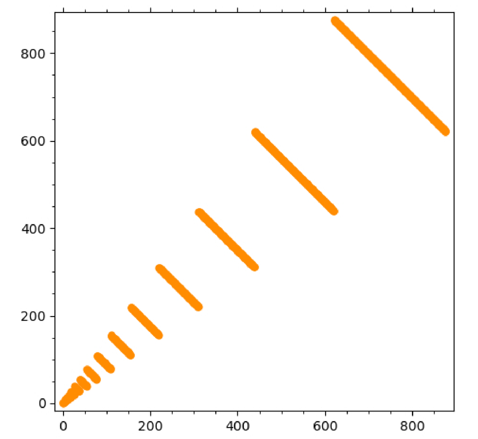
There are sequences of layered permutations $w(b^{(n)}) \in S_n$ so that

$$\lim_{n\to\infty} \frac{1}{n^2} \log_2 \mathfrak{G}_{w(b^{(n)})}(1^n) = \frac{1}{2}.$$

E.g. geometric $b_i \sim (1-\alpha)\alpha^{i-1} n$ for any $\alpha \in [1/\sqrt{2}, 1)$.

Example: maximal Grothendieck on layered

$\sum_{w \in S_n} \mathfrak{G}_w^1(1^n) = 2^{\binom{n}{2}} \implies \max_{w \in S_n} \mathfrak{G}_w^1(1^n) = 2^{\frac{1}{2}n^2 - O(n \log n)}$. $w = ?$



$\sum_{i=1}^k b_i$	b_1	$\frac{1}{n^2}\log_2 \mathfrak{G}_{w(b)}(1^n)$
13	4	0.328982
19	6	0.375242
27	8	0.406540
39	12	0.433385
55	16	0.450935
78	23	0.464651
110	32	0.474448
156	46	0.481823
220	64	0.486974
311	91	0.490735
439	128	0.493404
621	182	0.495329
877	256	0.496684

$b = (256, 182, 128, 91, 64, 46, 32, 23, 16, 12, 8, 6, 4, 3, 2, 2, 1, 1)$. Note: $b_i/b_{i+1} \approx 1/\sqrt{2}$

Idea of proof

For nonnegative integers k and n, we have

$$F(k, n-k) := \mathfrak{G}_{w(k,n)}^{\beta=1}(1, \ldots, 1) = 2^{-\binom{k}{2}} \det[s_{n-2+i+j}]_{i,j=1}^k,$$

where s_m, is the little Schröder numbers, which count paths from $(0,0)$ to $(2m, 0)$ with steps $(1,1), (1,-1), (2,0)$ that do not cross below and no $(2,0)$ steps on the x-axis . Probability bottom $n-k$ levels frozen to horizontal dominoes ($R_{k,n-k}$) via arctic circle theorem[a] for $k > n/\sqrt{2}$:

$\mathrm{Prob}(R_{k,n-k}) = 1 - e^{-O(n)}, \quad n \to \infty$

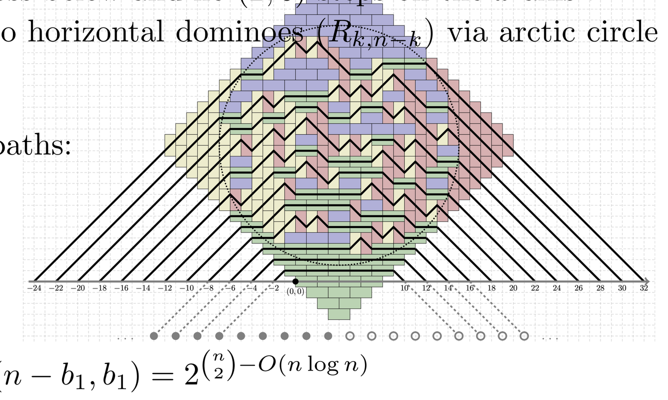
Probability via non-intersecting lattice paths:

$\mathrm{Prob}(R_{k,n-k}) = \frac{\det[s_{n-k-2+i+j}]_{i,j=1}^k}{2^{\binom{n}{2}}}$

$\mathfrak{G}_{u \times w}(1^n) = \mathfrak{G}_u(1^k) \cdot \mathfrak{G}_{\mathrm{id}_k \times w}(1^n)$.

$F(k, n) = 2^{\binom{n}{2} - \binom{k}{2} - O(n)}$

$\mathfrak{G}_{w(\ldots,b_2,b_1)}(1^n) = \mathfrak{G}_{w(\ldots,b_3,b_2)}(1^{n-b_1}) \cdot F(n-b_1, b_1) = 2^{\binom{n}{2} - O(n \log n)}$



[a] [Jockush-Propp-Shor'98], [Cohn-Elkies-Propp'96], [Cohn-Kenyon-Propp'01]

Main result II Asymptotics of typical Grothendieck permutation

1. There exists a limiting height function h° such that

$$\lim_{n\to\infty} n^{-1} H(\lfloor nx \rfloor, \lfloor ny \rfloor) = \mathsf{h}^\circ(\mathsf{x}, \mathsf{y}), \qquad (\mathsf{x}, \mathsf{y}) \in [0,1]^2,$$

the convergence is in probability. The function $\mathsf{h}^\circ(\mathsf{x}, \mathsf{y})$ is explicit, continuous, depends only on p.

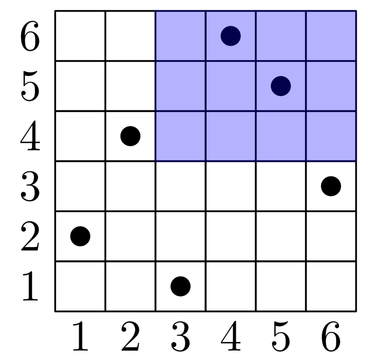
2. The fluctuations of $H(x, y)$ around h° belong to the KPZ universality class:

$$\lim_{n\to\infty} \mathrm{Prob}\left(\frac{H(\lfloor nx \rfloor, \lfloor ny \rfloor) - n\mathsf{h}^\circ(\mathsf{x}, \mathsf{y})}{\mathsf{v}(\mathsf{x}, \mathsf{y}) n^{1/3}} \le r \right) = F_2(r), \qquad r \in \mathbb{R},$$

F_2 is the cumulative distribution function of the Tracy–Widom GUE distribution[a], and $\mathsf{v}(\mathsf{x}, \mathsf{y})$ is explicit.

[a] = longest increasing subsequence in uniform permutations

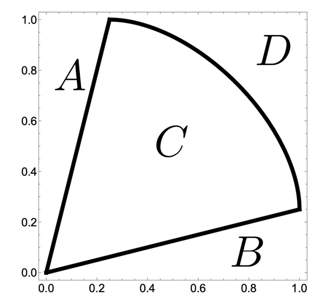
$H(x, y) := |\{w^{-1}(x), \ldots, w^{-1}(n)\} \cap \{y, y+1, \ldots, n\}|$

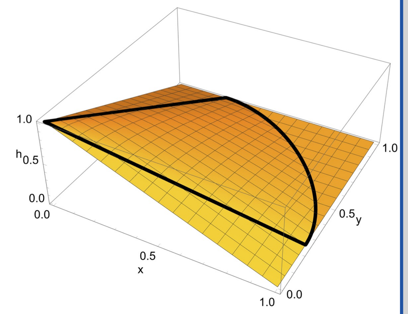


$H(3, 4) = 2$

The limit shape h°

$$\mathsf{h}^\circ(\mathsf{x}, \mathsf{y}) = \begin{cases} 1 - \mathsf{y}, & (\mathsf{x}, \mathsf{y}) \in A := \left\{0 < \mathsf{x} < 1 - p,\ \frac{\mathsf{x}}{1-p} < \mathsf{y} < 1\right\}; \\ 1 - \mathsf{x}, & (\mathsf{x}, \mathsf{y}) \in B := \{0 < \mathsf{x} < 1,\ 0 < \mathsf{y} < (1-p)\mathsf{x}\}; \\ 1 + \frac{2}{p}\sqrt{(1-p)\mathsf{x}\mathsf{y}} - \frac{\mathsf{x}+\mathsf{y}}{p}, & (\mathsf{x}, \mathsf{y}) \in C := \left\{ \begin{array}{l} 0 < \mathsf{x} < 1,\ (1-p)\mathsf{x} < \mathsf{y} < \frac{\mathsf{x}}{1-p}, \\ (\mathsf{x}, \mathsf{y}) \text{ below } E_p \end{array} \right\}; \\ 0, & (\mathsf{x}, \mathsf{y}) \in D := \{(\mathsf{x}, \mathsf{y}) \text{ above } E_p\}. \end{cases}$$





Delta mass γ_p on $E_p := \left\{(\mathsf{x}, \mathsf{y}) : \frac{(\mathsf{y}-\mathsf{x})^2}{p} + \frac{(\mathsf{y}+\mathsf{x}-1)^2}{(1-p)} = 1,\ 1-p \le \mathsf{x} \le 1\right\} \subset [0,1]^2$.

Colored models

$$w_p(a, b; c, d) = w_p\big({}_b \overset{c}{\underset{a}{+}} {}_d \big),$$

Weights ($0 \le a < b \le n$):

$$w_p(a, a; a, a) = 1;\ w_p(b, a; b, a) = p, w_p(b, a; a, b) = 1 - p;$$

$$w_p(a, b; a, b) = 0, w_p(a, b; b, a) = 1,$$

The random permutation $\mathbf{w} \in S_n$ obtained from this colored stochastic six-vertex model has the same distribution as the Grothendieck random permutation.

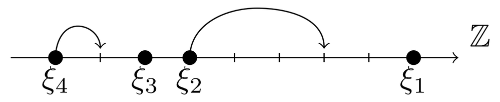
$$H(x, y) = \sum_{i=x}^n \sum_{j=y}^n \mathbf{1}_{\mathbf{w}^{-1}(i)=j}$$

Uncolored stochastic vertex model with empty (0) or not (1) pipes:

$$w_p^\bullet(0,0;0,0) = 1, \quad w_p^\bullet(1,1;1,1) = 1, \quad w_p^\bullet(1,0;1,0) = p,$$
$$w_p^\bullet(1,0;0,1) = 1-p, \quad w_p^\bullet(0,1;0,1) = 0, \quad w_p^\bullet(0,1;1,0) = 1.$$

Fix $1 \le x \le n$. In the colored stochastic vertex model erase all pipes of colors $< x$, and identify all the remaining pipes for colors $\ge x$. The resulting random configuration of uncolored pipes evolves according to the color-blind stochastic vertex model $w_p^\bullet$.

Vertex models (pipes) to TASEP

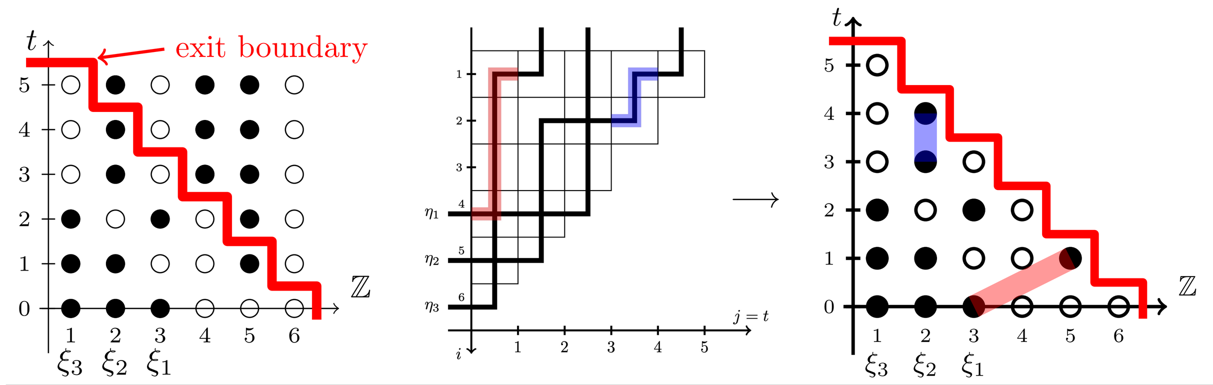


$\xi(t) := (\xi_1(t) > \ldots > \xi_k(t)) \subset \mathbb{Z}$

$$\xi_i(t+1) = \xi_i(t) + \min\left(G_i(t+1), \xi_{i-1}(t) - \xi_i(t) - 1\right), \qquad 1 \le i \le k,$$

$$\mathrm{Prob}\left(G_i(t+1) = m\right) = (1-p)p^m, \qquad m \in \mathbb{Z}_{\ge 0}.$$

$T_{\mathrm{exit}}(i) := \min\{t : \xi_i(t) \ge n + 1 - t\}$. $1 \le T_{\mathrm{exit}}(1) < T_{\mathrm{exit}}(2) < \ldots < T_{\mathrm{exit}}(k) \le n$.

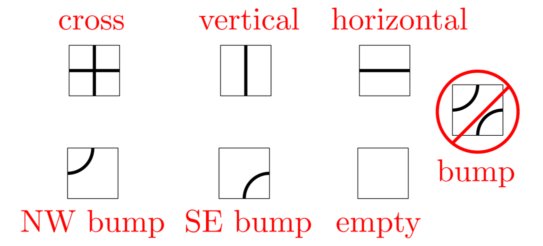


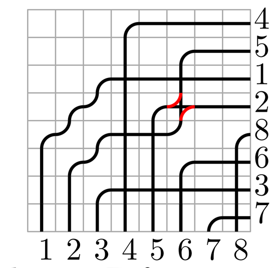
Theorem: For any $1 \le x, y \le n$ and $0 \le h \le n - x + 1$, we have

$$\mathrm{Prob}_{\mathbf{w}}\left(H(x, y) \le h\right) = \mathrm{Prob}_{\mathrm{TASEP}}\left(T_{\mathrm{exit}}(n - x + 1 - h) \le y - 1\right)$$
$$= \mathrm{Prob}_{\mathrm{TASEP}}\left(\xi_{n-x+1-h}(y-1) \ge n - y + 2\right).$$

Background: Grothendieck via 6-vertex model

A **bumpless pipe dream (BPD)** is a tiling D of the $n \times n$ square with six types of tiles:





bumpless pipe dream D for $w = 45128637$ (after resolving double crossings)

Theorem (Weigandt 2019)

For any permutation $w \in S_n$, we have

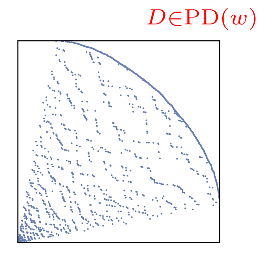
$$\mathfrak{G}_w^\beta(x_1, \ldots, x_n) = \beta^{-\ell(w)} \sum_{D \in \mathrm{BPD}(w)} \prod_{(k,l) \in \mathrm{emptytile}(D)} \beta x_k \prod_{(k,l) \in \mathrm{NWbump}(D)} (1 + \beta x_k).$$

The typical Grothendieck permutation

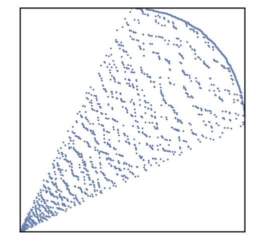
cross ┼ with probability p, ╯╭ with probability $1 - p$.

- Resolve after first intersection (Demazure product)

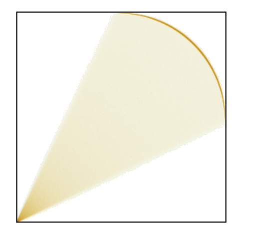
$$\mathrm{Prob}(\mathbf{w} = w) = \sum_{D \in \mathrm{PD}(w)} p^{\mathrm{cross}(D)}(1-p)^{\mathrm{elbow}(D)} = (1-p)^{\binom{n}{2}} \mathfrak{G}_w^{\beta=1}\left(\frac{p}{1-p}, \ldots, \frac{p}{1-p}\right),$$



$n = 2000$, $p = \frac{4}{5}$,



$p = \frac{1}{2}$,



the permuton

Application: inversions

Let $\mathbf{w} = \mathbf{w}(n) \in S_n$ be the Grothendieck random permutations with a fixed parameter $p \in (0, 1)$. Then

$$\lim_{n\to\infty} \frac{\mathrm{inv}(\mathbf{w}(n))}{\binom{n}{2}} = \gamma_p := 1 - \sqrt{\frac{1-p}{p}} \arccos \sqrt{1-p}.$$

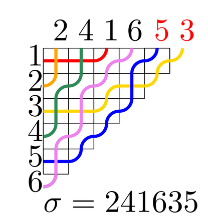
In the original model we have $\gamma_{\frac{1}{2}} = 1 - \frac{\pi}{4}$.

The natural model (Defant 2024 at Stanley80)

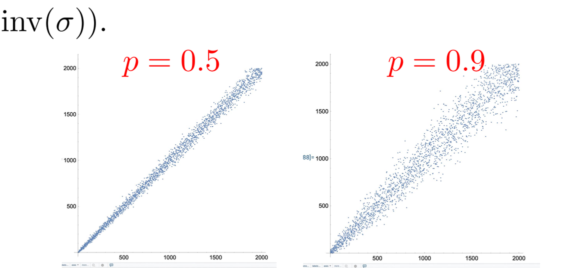
cross ┼ with probability p, ╯╭ with probability $1 - p$.

Let σ be the resulting permutation.

Problem: study $\mathbb{E}(\mathrm{inv}(\sigma))$ and $\mathrm{Var}(\mathrm{inv}(\sigma))$.



$\sigma = 241635$



- $\mathrm{dis}(w) := \sum_{i=1}^n |i - w_i|$

Diaconis-Graham: $\frac{1}{2}\mathrm{dis}(w) \le \mathrm{inv}(w) \le \mathrm{dis}(w)$

Theorem Fix $p \in [0, 1)$. For every $\varepsilon > 0$ and sufficiently large n, we have

$$\frac{2}{3\sqrt{\pi}}(1-\varepsilon)n^{3/2}\sqrt{\frac{p}{1-p}} \le \mathbb{E}[\mathrm{inv}(\mathbf{w})] \le \frac{4}{3\sqrt{\pi}}(1+\varepsilon)n^{3/2}\sqrt{\frac{p}{1-p}}.$$

~~Conjecture~~ **Theorem (Defant 2024)** $\mathbb{E}[\mathrm{inv}(\mathbf{w})] = \frac{2\sqrt{2}}{3\sqrt{\pi}} n^{3/2} \sqrt{\frac{p}{1-p}}$

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Arxiv:2407.21653 and code