Generic pipe dreams, lower-upper varieties, and Schwartz–MacPherson classes

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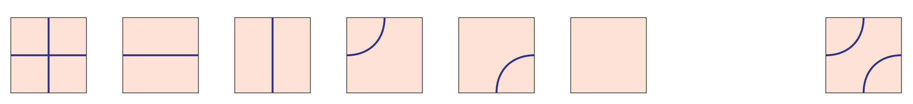
Overview of results

We introduce *generic pipe dreams*, and associate to each one a simple polynomial in $\mathbb{Z}[x_1,\ldots,x_n,y_1,\ldots,y_n,A,B]$. Summing these polynomials for a fixed permutation $\pi \in S_n$ results in a *generic pipe dream polynomial* G_π.

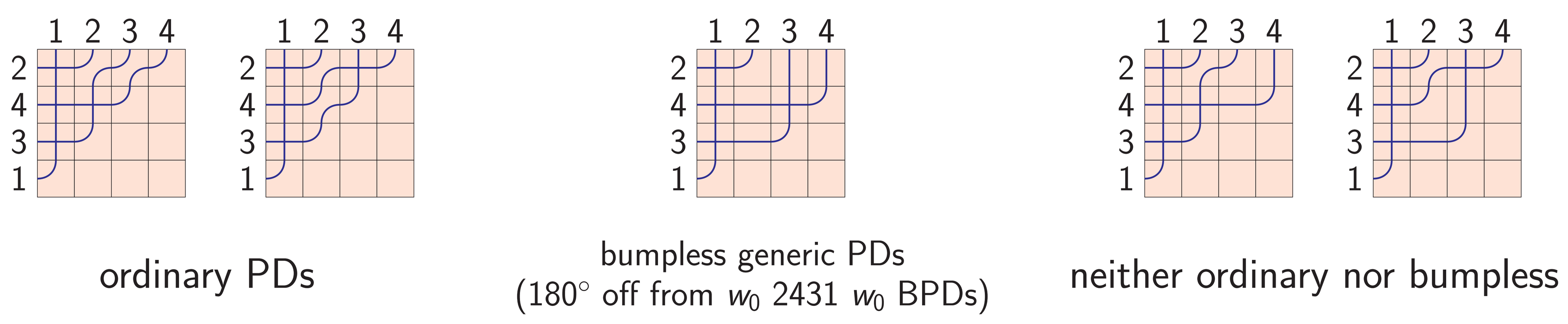
- We establish a recurrence on these polynomials, from which we prove that their A-leading and B-leading forms are double Schubert polynomials. Those forms recover the classic pipe dream and bumpless pipe dream formulæ for double Schubert polynomials (in particular, giving new proofs thereof).
- We show that G_π computes the equivariant cohomology class of the *lower-upper variety* E_π introduced in [K05]. Since E_π projects to the closed matrix Schubert variety $\overline{B_-\pi B_+} \subseteq M_n(\mathbb{C})$, this gives a new proof that $\left[\overline{B_-\pi B_+}\right]$ is the double Schubert polynomial.
- We show G_π computes the equivariant Chern–Schwartz–MacPherson class of the open matrix Schubert variety $B_-\pi B_+ \subseteq M_n(\mathbb{C})$, studied also in [RW22].

Combinatorics: generic pipe dreams

Define a **generic pipe dream tile** as any of the following:



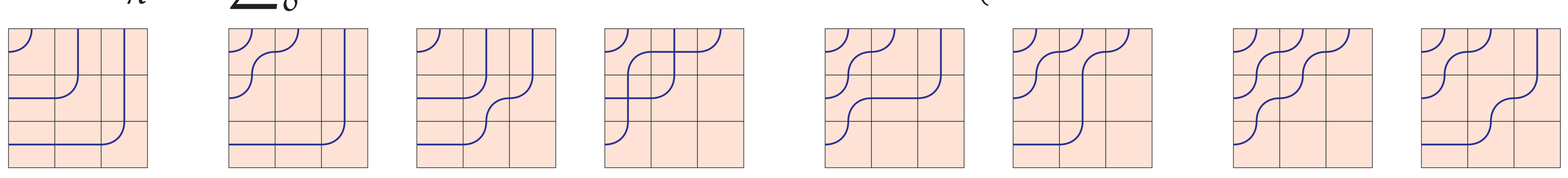
Each pipe will carry a distinct label, generally from $[n] := \{1,\ldots,n\}$. We assemble these tiles into $n\times n$ squares with blank labels on East and South, calling them **generic pipe dreams** or **GPDs**. For $\pi = 2431$ they are these:



The **generic pipe dream polynomial** G_π is defined to be the sum $\sum_{\delta\in GPDs(\pi)} wt(\delta)$ over all GPDs δ for π, of a product of factors:

$$wt(\delta) := \prod_{i,j\in[n]} \begin{cases} A + x_i - y_j & \text{if (cross, vertical, horizontal tile) at } (i,j) \\ B - x_i + y_j & \text{if (blank tile) at } (i,j) \\ A + B & \text{otherwise} \end{cases}$$

We take particular interest in the specialization $x_i, y_i \mapsto 0$, $A, B \mapsto 1$, in which case $G_\pi \mapsto \sum_\delta 2^{\#\text{(bump tiles)}}$. For $\pi = 123$ this is $2^3(1+2+2+2+4+4+8+8)$:



Geometry: lower-upper varieties

The **lower-upper scheme** $E := \{(X,Y) \in M_n(\mathbb{C})^2\colon XY \text{ lower triangular, } YX \text{ upper}\}$ [K05] has an action of $B_- \times B_+$ by $(b,c)\cdot(X,Y) := (bXc^{-1}, cYb^{-1})$. Its $n!$ components, one for each $\pi \in S_n$, are shown in [K05] to be of the form

$$E_\pi := \overline{\{(X,Y)\in E\colon diag(XY) = \pi\cdot diag(YX) \text{ nonrepeating}\}} = \overline{(B_-\times B_+)\cdot\{(\pi,\pi^{-1}D)\colon D \text{ diagonal}\}}$$

E_1 is a degeneration of the **commuting scheme** $\{(X,Y)\colon XY = YX\}$ (the motivation for [K05]), potentially up to lower-dimensional components. Those don't affect the degree (and conjecturally, are not there at all).

Cohomology: CSM classes

To a subvariety $X \subseteq M$ of a smooth complex variety, one can associate a class $csm(X) \in H^*_{\text{dilation}}(T^*M) \cong H^*(M)[\hbar]$, uniquely determined by

1. $csm(A \amalg B \subseteq M) = csm(A \subseteq M) + csm(B \subseteq M)$
2. $csm(M \subseteq M) = [M \subseteq T^*M] = e(T^*M)$, the dilation-equivariant Euler class
3. good properties under pushforward, which involve constructible functions and "integration w.r.t. Euler characteristic measure".

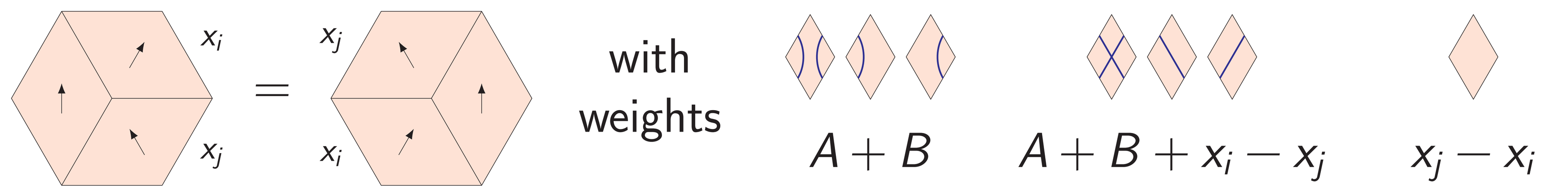
As $\hbar \to \infty$ we recover $\left[\overline{X}\right]$ as the leading term.
If T acts on X and M then $csm(X \subseteq M)$ can be defined in $H^*_{T\times\text{dilation}}(T^*M)$.
Using $\mathcal{D}$-modules, X gets a "characteristic cycle" in T^*M with class $\pm csm(X)$.

1. A recurrence

Theorem 1. Given $\pi \in \mathcal{S}_n$, define $\pi' = \pi r_i$ by switching $\pi(i)$ and $\pi(i+1)$. Then

$$G_\pi = \frac{1}{x_i - x_{i+1}}\Big((A+B)G_{\pi'} - (A+B+x_i-x_{i+1})r_i G_{\pi'}\Big)$$

Proof ingredient. This is based on the Yang–Baxter equation



2. Leading forms

Theorem 2. $S_\pi := \lim_{A\to\infty} A^{\ell(\pi)-n^2} G_\pi$ is π's double Schubert polynomial. Similarly, $S_\pi(-y_n,\ldots,-y_1,-x_n,\ldots,-x_1) = \lim_{B\to\infty} B^{\ell(\pi)-n^2} G_{w_0\pi w_0}$.

Proof. Taking $A \to \infty$ in the recurrence of Theorem 1 recovers the BGG recurrence on double Schubert polynomials. The $B\to\infty$ story is similar.

In addition, these limits pick out certain GPDs as A or B goes to ∞:
B-leading form. If two pipes cross twice, replace each (cross tile) with (bump tile) to increase the power of B. If some pipe goes into the SE triangle, then it crosses an invisible pipe twice; now do the same trick. Those GPDs that survive are classic PDs.
A-leading form. The total length, in # of squares traversed, of pipe i is $i + \pi^{-1}(i) - 1$; summing over i we get $\binom{n+1}{2} + \binom{n+1}{2} - n = n^2$. Hence each tile used contains one visible pipe, on average. Some tiles accommodate two visible pipes, some zero, so #(cross) + #(bump) = #(blank). We want to minimize the number of (blank), as those don't admit an A term, so we must minimize the number of crosses (to $\ell(\pi)$, using no double crosses) and bumps (to zero). Those GPDs that survive are bumpless PDs (180° rotated from standard).

3. Fundamental classes of lower-upper varieties

Theorem 3. Write $H^*_{B_-\times B_+\times(\mathbb{C}^\times)^2} = \mathbb{Z}[x_1,\ldots,x_n,y_1,\ldots,y_n,A,B]$ and let $(\mathbb{C}^\times)^2 \circlearrowright M_n(\mathbb{C})^2$ by scaling each factor. Then $[E_\pi \subseteq M_n(\mathbb{C})^2] = (A+B)^{-n} G_\pi$.

Proof idea. $[E_{w_0}] = (A+B)^{-n} G_{w_0}$, easy to show since E_{w_0} is a linear subspace and G_{w_0} is a sum over a single GPD. Then we show with geometric techniques either (1) that the $[E_\pi]$ satisfy the recurrence from theorem 1, or (2) that E_π degenerates to $\bigcup_{\delta\in GPDs(\pi)} F_\delta$ up to embedded components, where F_δ is a quadratic complete intersection with $[F_\delta] = wt(\delta)$.

Corollary. The degree of the nth commuting scheme is $G_{123\cdots n}|_{x_i,y_i\mapsto 0,\ A,B\mapsto 1}$ times 2^{-n}, e.g. for $n=3$ the degree is $31 = 1+2+2+2+4+4+8+8$.

4. CSM classes of open matrix Schubert varieties

Theorem 4.

1. For $\rho \in S_{2n}$, the space $Y_\rho := \left\{ M \in M_n(\mathbb{C})\colon \begin{bmatrix} M & I \\ I & 0 \end{bmatrix} \in B_-^{2n}\rho B_+^{2n} \right\}$ has CSM class given by a sum over $n\times n$ pipe dreams with only (bump, cross) tiles, no blank labels. This space Y_ρ is characterized by the Northwest ranks of M, its Southeast ranks, the ranks of consecutive rows, and of consecutive columns.
2. $B_-\pi B_+ = \coprod_{\sigma\in S_n} Y_{\pi\oplus\sigma}$
3. $csm(B_-\pi B_+ \subseteq M_n(\mathbb{C})) = G_\pi|_{A=0,B=-\hbar}$
4. $E_\pi \cap \{diag(XY) = 0\}$ gives the "characteristic cycle" in T^*M_n of $B_-\pi B_+$. In particular, this is "why" theorems 3 and 4(3) agree up to $(A+B)^n$.

Proof ideas. 1. CSM classes of Bruhat cells match the Maulik–Okounkov stable classes computed in [S17]. (More translation is necessary beyond that!)
2. The LHS only depends on NW ranks; the RHS is the union over all possible arrays of SE ranks. (The rank of each i rows or i columns is always i.)
3. Use part 1 and the additivity of CSM classes. Group the many, many terms based on the pipes from the North to the West. Each grouping has 2^k terms (for various k) that sum up to a single term from the formula for G_π.
4. The LHS lies inside the zero level set of the moment map $\Phi_{B_-\times B_+}$ so, is a conical Lagrangian like the RHS, whose class is $\pm$ the CSM class. Each side is $B_-\times B_+$-invariant hence supported on a union of conormal varieties to the $B_-\times B_+$-orbits. The classes of those are linearly independent over $\mathbb{Z}$.

References

[K05] Allen Knutson, "Some schemes related to the commuting variety" J. of Algebraic Geometry (2005)
[RW22] Piotr Rudnicki and Andrzej Weber, "Characteristic classes of Borel orbits of square-zero upper-triangular matrices" J. of Algebra (2022)
[S17] Changjian Su, "Restriction formula for stable basis of Springer resolution" Selecta Mathematica (2017)