Interlacing triangles, Schubert puzzles, and graph colorings

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Interlacing triangular arrays

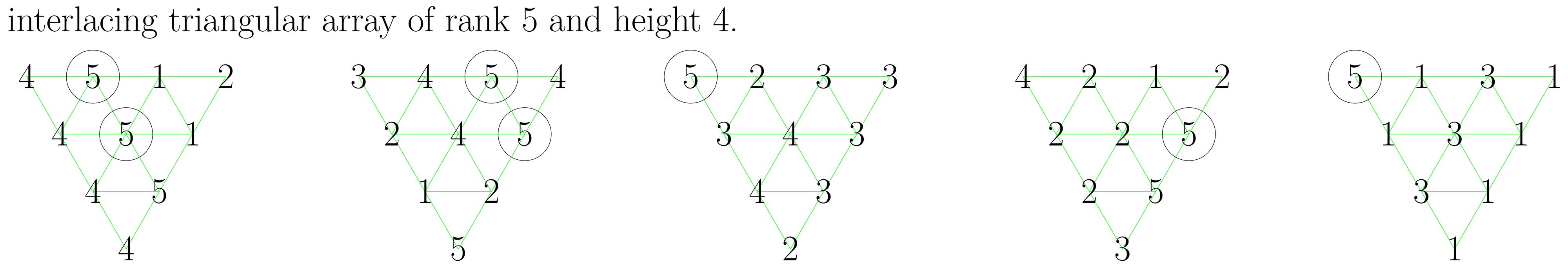
Interlacing triangular arrays are introduced by Aggarwal–Borodin–Wheeler [1] to study certain probability measures.

Definition [1]

An **interlacing triangular array** T of rank m and height n is a collection $\{T_{j,k}^{(i)} \mid 1 \leq i \leq m, 1 \leq j \leq k \leq n\}$ of positive integers from $[m] = \{1, 2, \ldots, m\}$, subject to the following conditions:

- For each $k = 1, \ldots, n$ we have an equality of multisets:
$$\{T_{j,k}^{(i)} \mid 1 \leq i \leq m, 1 \leq j \leq k\} = \{1^k\} \cup \cdots \cup \{m^k\}.$$
- Let the horizontal coordinate of $T_{j,k}^{(i)}$ be $h(i,j,k) := in + j - (n+k)/2$. If $T_{j,k}^{(i)} = T_{j',k}^{(i')} = a$ for some i, j, i', j', k with $h(i,j,k) < h(i',j',k)$, then there must exist i'', j'' with $T_{j'',k-1}^{(i'')} = a$ and $h(i,j,k) < h(i'',j'',k-1) < h(i',j',k)$. This entry $T_{j'',k-1}^{(i'')}$ is said to *interlace* with $T_{j,k}^{(i)}$ and $T_{j',k}^{(i')}$.

Here is an interlacing triangular array of rank 5 and height 4.



Bijections

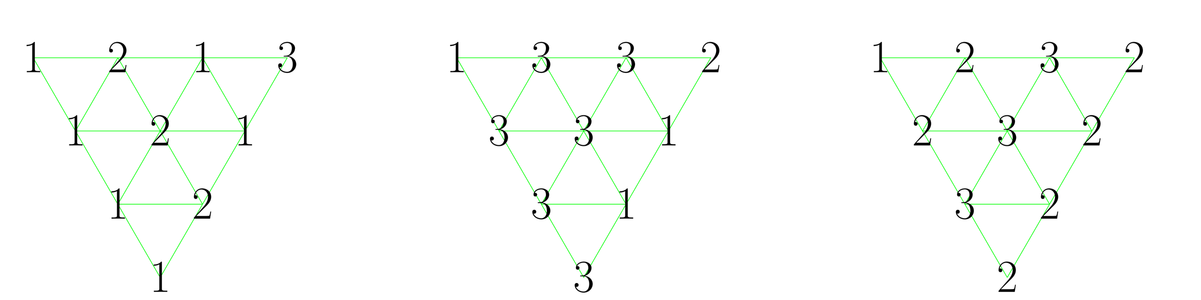
Theorem

The following sets of objects are in bijection with each other (with certain boundary conditions):

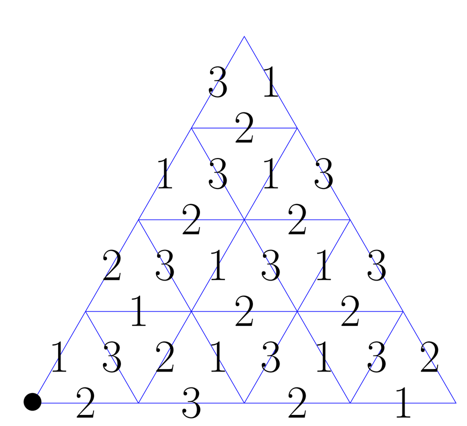
1. interlacing triangular arrays $\mathcal{T}_{3,n}$ of rank 3 and height n (with top row $\boldsymbol{\lambda}$);
2. 1/2/3-puzzles of size n, i.e. proper edge 3-colorings of the triangular grid Δ_n (with boundary $\boldsymbol{\lambda}$);
3. proper vertex 4-colorings of the triangular grid Δ_n with one fixed color (with boundary $\mathrm{col}(\boldsymbol{\lambda})$).

Can you figure out the bijections?

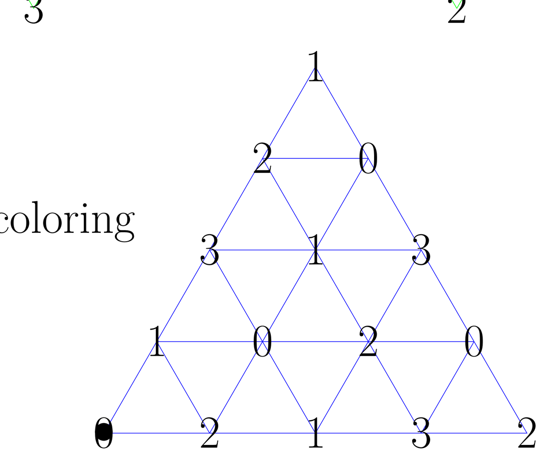
An interlacing triangular array:



An 1/2/3 puzzle:



A proper vertex 4-coloring with 0 at ●



With the bijections, we prove Conjecture A.3 of Aggarwal-Borodin-Wheeler [1].

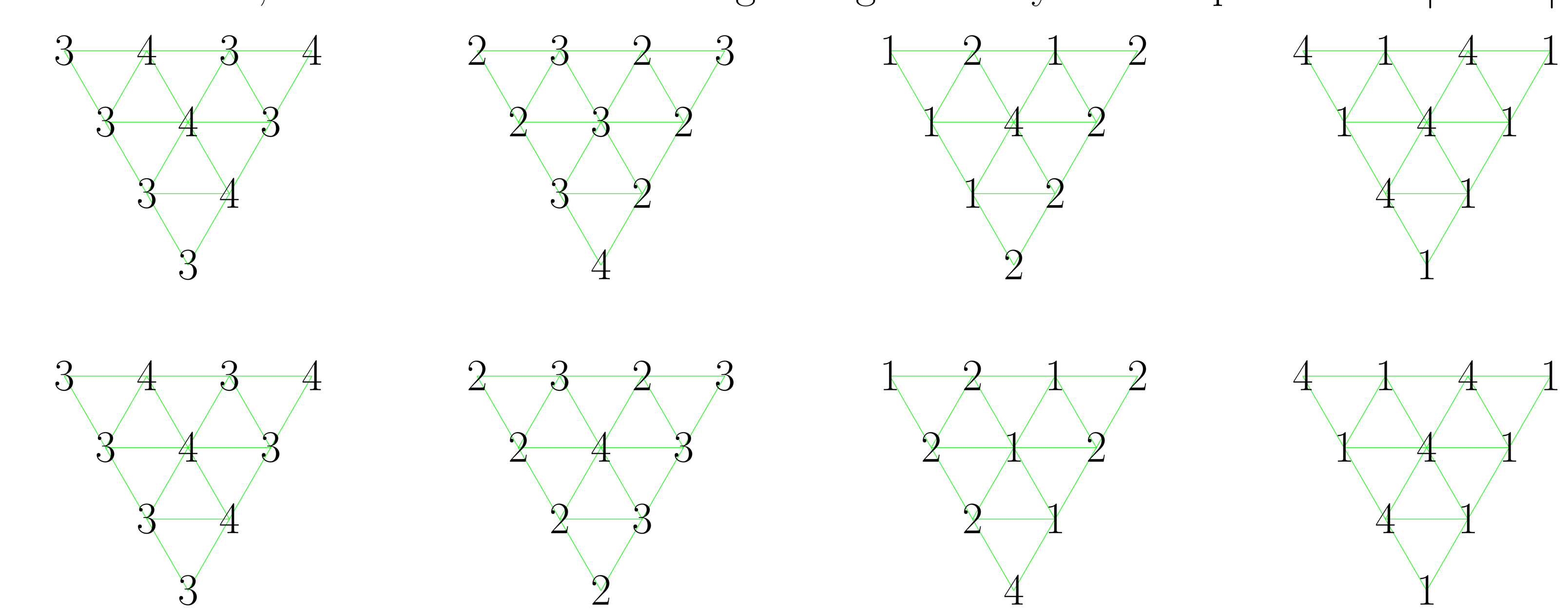
Interlacing triangular arrays compute structure constants

For a partition $\lambda \subset k \times (n-k)$, we can encode it via a lattice path, denoted $\lambda(a < b)$, using k a's (vertical steps) and $(n-k)$ b's (horizontal steps). Write s_λ for its **Schur function**. Write $\lambda^\vee$ for the **complement** of λ inside $k \times (n-k)$.
Denote the set of interlacing triangular arrays of rank m and height n whose top row is $(A_1, \ldots, A_m)$ by $\mathcal{T}_{m,n}(A_1, \ldots, A_m)$.

Theorem

Let $\lambda^{(1)}, \ldots, \lambda^{(m-1)}, \lambda$ be partitions in $k \times (n-k)$ such that $|\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(m-1)}|$. Then the coefficient of s_λ in the product $s_{\lambda^{(1)}} \cdots s_{\lambda^{(m-1)}}$ equals
$$|\mathcal{T}_{m,n}(\lambda^{(1)}(m-1 < m), \lambda^{(2)}(m-2 < m-1), \ldots, \lambda^{(m-1)}(1 < 2), \lambda^\vee(m < 1))|.$$

Let $k = 2$ and $n = 4$. We compute the coefficient of $s_{\boxplus}$ in $s_{\square}^3$, which is also the coefficient of $s_{\boxplus}$ in $s_{\square}^4$ by duality.
For $\lambda = \square$, $\lambda(a < b) = abab$. Thus, we need to find interlacing triangular arrays with top row 3434 | 2323 | 1212 | 4141:



Our main theorem generalizes to computing the structure constant in the K-theory $K(\mathrm{Gr}(k,n))$ with respect to the classes of the structure sheaf of the Schubert varieties (represented by the Grassmannian Grothendieck polynomials), and with respect to their duals, the ideal sheaves (represented by the dual Grothendieck polynomials). Our main theorem also generalizes to certain m-fold product in the ordinary cohomology of a partial flag variety (still within the known realm of the separated descent cases).

Enumeration and coloring

Let $\mathcal{T}_{m,n}$ be the set of interlacing triangular arrays of rank m and height n.
It is easy to show that $|\mathcal{T}_{2,n}| = 2^n$. And we have also seen enumerative properties of $|\mathcal{T}_{3,n}|$ via bijections.

Conjecture

$|\mathcal{T}_{4,n}|$ equals the number of edge colorings of the square grid using 4 colors such that for each square, all four sides are different.

What about $m \geq 5$?

References

[1] Amol Aggarwal, Alexei Borodin, and Michael Wheeler.
Coloured corner processes from asymptotics of LLT polynomials.
Adv. Math., 451:109781, 2024.

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Interlacing triangles, Schubert puzzles, and graph colorings.
Comm. Math. Phys., 406(5):Paper No. 118, 24, 2025.

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