RSK as a linear operator

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The Robinson–Schensted–Knuth correspondence (RSK) is a bijection between nonnegative integer matrices and pairs of Young tableaux. We study it as a linear operator on the coordinate ring of matrices, proving results about its diagonalizability, eigenvalues, trace, and determinant. Our criterion for diagonalizability involves the ADE classification of Dynkin diagrams, as well as the diagram for E_9. Preprint available at arXiv:2410.23009 (or via the QR code).

Introduction

Robinson, Schensted, and Knuth
Uncovered a beautiful truth
From a matrix they'd go
To pairs of tableaux
And back with scarcely an oof!

To classify with ADE
Is common in fields close to Lie
Yet it also (we say)
Tells when RSK
Can diagonalizable be!

RSK is a bijection between two natural bases of the coordinate ring $\mathbb{C}[\mathsf{Mat}_{m,n}]$ of $m \times n$ matrices:

The "obvious" *monomial basis*, e.g.

$$z_{12}z_{23}z_{31} \leftrightarrow \begin{bmatrix} 0&1&0\\ 0&0&1\\ 1&0&0 \end{bmatrix}$$

The "representation-theoretic" *bitableaux basis*, e.g.

$$\left(\ytableaushort{12,3}\;\middle|\;\ytableaushort{13,2}\right) \leftrightarrow \begin{vmatrix} z_{11} & z_{12}\\ z_{31} & z_{32} \end{vmatrix} \begin{vmatrix} z_{23} \end{vmatrix}.$$

This bijection extends to define a linear operator on $\mathbb{C}[\mathsf{Mat}_{m,n}]$. For example, on the *weight subspace* indexed by $(\sigma, \pi) = (111, 111)$, $\mathbb{C}[\mathsf{Mat}_{3,3}]$ has bases

$$\{z_{11}z_{22}z_{33}, z_{11}z_{32}z_{23}, z_{21}z_{12}z_{33}, z_{21}z_{32}z_{13}, z_{31}z_{12}z_{23}, z_{31}z_{22}z_{13}\}$$

and

$$\left\{\left(\ytableaushort{123}, \ytableaushort{123}\right), \left(\ytableaushort{12,3}, \ytableaushort{12,3}\right), \left(\ytableaushort{13,2}, \ytableaushort{13,2}\right), \left(\ytableaushort{13,2}, \ytableaushort{12,3}\right), \left(\ytableaushort{12,3}, \ytableaushort{13,2}\right), \left(\ytableaushort{1,2,3}, \ytableaushort{1,2,3}\right)\right\}$$

$$= \left\{ z_{11}z_{22}z_{33}, \begin{vmatrix} z_{11} & z_{13}\\ z_{31} & z_{33}\end{vmatrix} z_{22}, \begin{vmatrix} z_{11} & z_{12}\\ z_{21} & z_{22}\end{vmatrix} z_{33}, \begin{vmatrix} z_{11} & z_{13}\\ z_{21} & z_{23}\end{vmatrix} z_{32}, \begin{vmatrix} z_{11} & z_{12}\\ z_{31} & z_{32}\end{vmatrix} z_{23}, \begin{vmatrix} z_{11} & z_{12} & z_{13}\\ z_{21} & z_{22} & z_{23}\\ z_{31} & z_{32} & z_{33}\end{vmatrix} \right\}.$$

RSK preserves the ordering of these bases as written. Thus the matrix representing the operator is

$$\mathsf{RSK}_{111,111} = \begin{bmatrix} 1&1&1&0&0&1\\ 0&0&0&1&1&-1\\ 0&0&-1&0&0&-1\\ 0&0&0&-1&0&1\\ 0&0&0&0&-1&1\\ 0&-1&0&0&0&-1 \end{bmatrix}.$$

This matrix is diagonalizable, with characteristic polynomial $p_{\mathsf{RSK}_{111,111}}(t) = (t-1)(t+1)^2(t^3+2t^2+1)$.

Eigenvalues

Theorem. *All dth roots of unity are eigenvalues of $\mathsf{RSK}_{m,n,d}$ if $m \geq 2$ and $n, k \geq d$.*
Theorem. *The characteristic polynomial of $\mathsf{RSK}_{m,n,d}$ is not solvable by radicals whenever $m, n \geq 3$ and $d \geq 4$.*

Determinant and Trace

Theorem. *Fix d and let $2^r > d$. Then $\det \mathsf{RSK}_{m,n,d}$ has period 2^r in both m and n:*

$$\det \mathsf{RSK}_{m,n,d} = \det \mathsf{RSK}_{m+2^r,n,d} = \det \mathsf{RSK}_{m,n+2^r,d}.$$

Theorem. *For fixed d, the trace of $\mathsf{RSK}_{m,n,d}$ is a polynomial in $O(m^d n^d)$.*

$m \backslash d$	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	−1	1	−1	1	1	1	1	1
3	1	−1	−1	1	1	−1	−1	⋯	
4	1	1	1	1	1	⋯			
5	1	1	1	1	⋯				

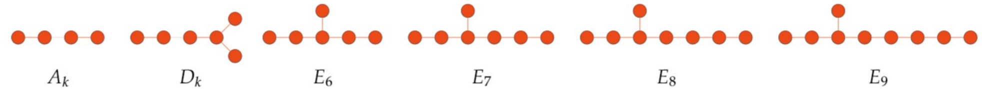
Table 1: *Values of* $\det \mathsf{RSK}_{m,m,d}$

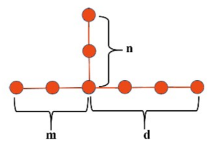
$m \backslash d$	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	4	8	12	17	24	32	40	49	60
3	9	27	42	70	160	241	203	⋯	
4	16	64	48	−33	613	1107	⋯		
5	25	125	−175	−1650	2853	⋯			

Table 2: *Values of* $\mathrm{Tr}\ \mathsf{RSK}_{m,m,d}$

Diagonalizability

Theorem. *The matrix $\mathsf{RSK}_{m,n,d}$ is diagonalizable if and only if $d \leq 3$, or the graph associated to (m, n, d) is one of the following Dynkin diagrams:*





The graph associated to (m, n, d).

$$\mathsf{RSK}_{211,211} = \begin{bmatrix} 1&1&1&0&0&1&1\\ 0&0&0&1&1&-1&0\\ 0&0&-1&0&0&-1&-1\\ 0&0&0&-1&0&1&0\\ 0&0&0&0&-1&1&0\\ 0&-1&0&0&0&-1&-1\\ 0&0&0&0&0&0&1 \end{bmatrix},$$

A minimal weight space where RSK is not diagonalizable. The eigenvalue 1 has algebraic multiplicity 2 but geometric multiplicity 1.

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