A GENERALIZATION OF DEODHAR'S DEFECT STATISTIC FOR IWAHORI–HECKE ALGEBRAS OF TYPE BC

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HYPEROCTAHEDRAL GROUP $\mathfrak{B}_n$

$\mathfrak{B}_n :=$ permutations $w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n$ of $\overline{n} \cdots \overline{2}\,\overline{1}\,1\,2 \cdots n$ with $w_{\overline{i}} = \overline{w_i}$.

- Generated by $s_0, s_1, \ldots, s_{n-1}$ with relations

$$\begin{aligned} {s_i}^2 &= e & i &= 0, \ldots, n-1, \\ s_i s_j s_i &= s_j s_i s_j & |i-j| &= 1, \quad i,j \geq 1, \\ s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0 & & \\ s_i s_j &= s_j s_i & |i-j| &\geq 2. \end{aligned}$$

- Inherited action on words $a_{\overline{n}} \cdots a_{\overline{1}} a_1 \cdots a_n$ is

s_i swaps letters in positions $i, i+1$ and $\overline{i}, \overline{i+1}$,
s_0 swaps letters in positions $\overline{1}, 1$.

- The parabolic subgroup of $\mathfrak{B}_n$ generated by $\{s_1, \ldots, s_{n-1}\}$ is $\mathfrak{S}_n$.

Reversals in $\mathfrak{B}_n$

- The **reversal** $s_{[a,b]}$ reverses letters $[a,b] := \{a, \ldots, b\} \smallsetminus \{0\}$, where $a = \overline{b}$ ($b > 0$) or $0 < a < b$.

Ex. $s_{[2,4]}(\overline{54321}12345) = \overline{52341}14325.$
$s_{[\overline{4},4]}(\overline{54321}12345) = \overline{5}4321\overline{1234}5.$

- For each reversal $s_{[a,b]} \in \mathfrak{B}_n$, we have

$$\widetilde{C}_{s_{[a,b]}}(q) = \sum_{v \leq s_{[a,b]}} T_v.$$

TYPE-BC HECKE ALGEBRA $H_n^{\mathrm{BC}}(q)$

Generated over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by $T_{s_0}, T_{s_1}, \ldots, T_{s_{n-1}}$ with relations

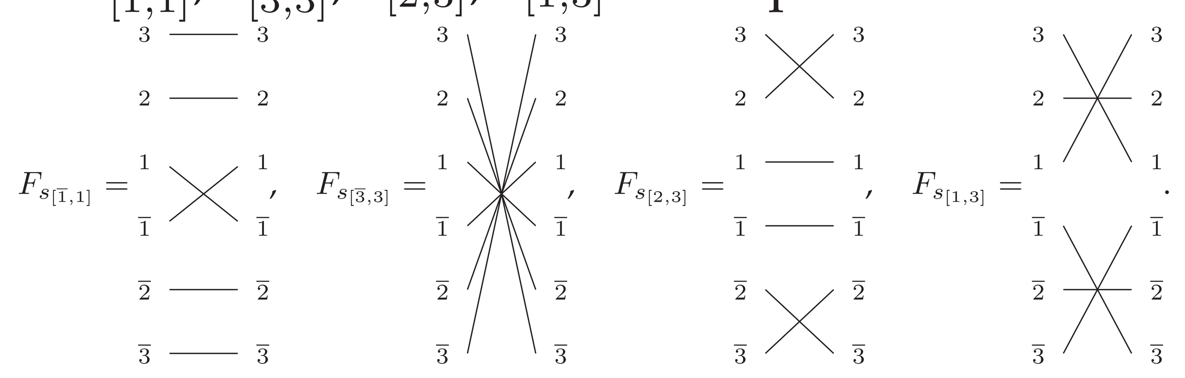
$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_e & i &= 0, \ldots, n-1, \\ T_{s_i} T_{s_j} T_{s_i} &= T_{s_j} T_{s_i} T_{s_j} & |i-j| &= 1, \\ T_{s_0} T_{s_1} T_{s_0} T_{s_1} &= T_{s_1} T_{s_0} T_{s_1} T_{s_0} & & \\ T_{s_i} T_{s_j} &= T_{s_j} T_{s_i} & |i-j| &\geq 2. \end{aligned}$$

- $H_n^{\mathrm{BC}}(1) \cong \mathbb{Z}[\mathfrak{B}_n]$
- Natural basis $\{T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}} \,|\, w = s_{i_1} \cdots s_{i_\ell} \text{ reduced in } \mathfrak{S}_n\}$.
- (Modified) Kazhdan-Lusztig basis $\{\widetilde{C}_w(q) \,|\, w \in \mathfrak{S}_n\}$, where $\widetilde{C}_w(q) = \sum_{v \leq w} P_{v,w}(q) T_v$.

STAR NETWORKS F

- A type-BC **simple star network** $F_{s_{[a,b]}}$ is the planar network corresponding to $s_{[a,b]} \in \mathfrak{B}_n$.

Ex. $s_{[\overline{1},1]}, s_{[\overline{3},3]}, s_{[2,3]}, s_{[1,3]}$ correspond to

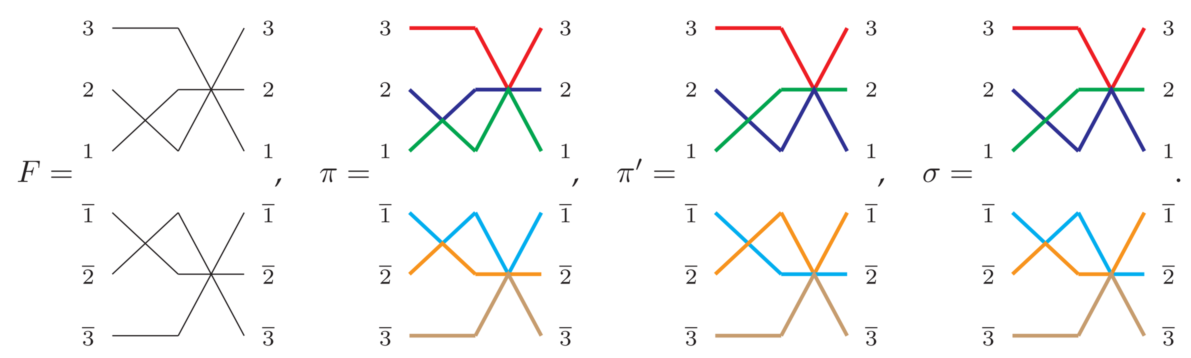


- A type-BC **star network** $F = F_{s_{[a_1,b_1]}} \circ \cdots \circ F_{s_{[a_k,b_k]}}$ is a concatenation of type-BC simple star networks.

PATH FAMILIES $\Pi^{\mathrm{BC}}(F)$

Call $\pi = (\pi_{\overline{n}}, \ldots, \pi_n)$ a type-BC **path family in F** when:
(1) π_i is a path from source i on left to some sink on right,
(2) π covers all edges of F, and
(3) π_i is a horizontal reflection of $\pi_{\overline{i}}$ for all $i = 1, \ldots, n$.

Ex. Consider the following path families in $F = F_{s_{[1,2]}} \circ F_{s_{[1,3]}}$

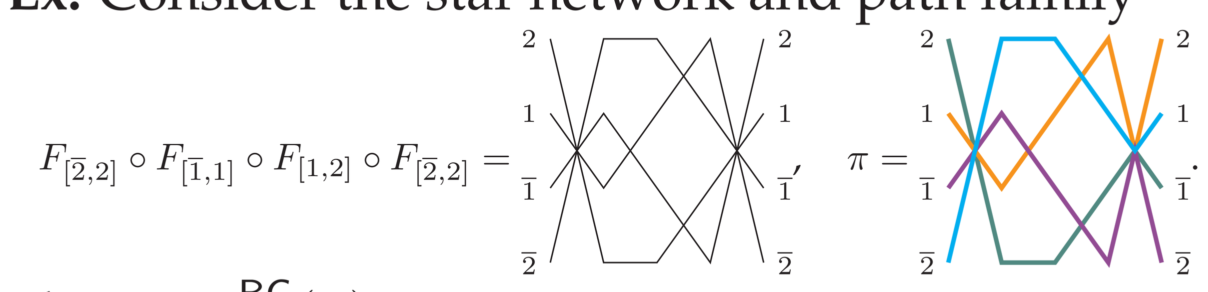


Then,
- $\pi, \pi' \in \Pi^{\mathrm{BC}}(F)$ with $\mathrm{type}(\pi) = e$, $\mathrm{type}(\pi') = s_1$.
- $\sigma \notin \Pi^{\mathrm{BC}}(F)$.

DEFECT NUMBER $\mathrm{dfct}^{\mathrm{BC}}(\pi)$

Given $\pi \in \Pi^{\mathrm{BC}}(F)$, define a type-BC **defect** of π to be a triple (π_i, π_j, k) with
(1) $|i| \leq j$, and
(2) π_i and π_j meet at an internal vertex of $F_{[c_k,d_k]}$ after having crossed an odd number of times.

Ex. Consider the star network and path family



Then $\mathrm{dfct}^{\mathrm{BC}}(\pi) = 4$.

MAIN RESULT

Theorem. Fix a sequence $(s_{[a_1,b_1]}, \ldots, s_{[a_k,b_k]})$ of reversals in $\mathfrak{B}_n$ and define the type-BC star network $F = F_{s_{[a_1,b_1]}} \circ \ldots \circ F_{s_{[a_k,b_k]}}$. Then we have

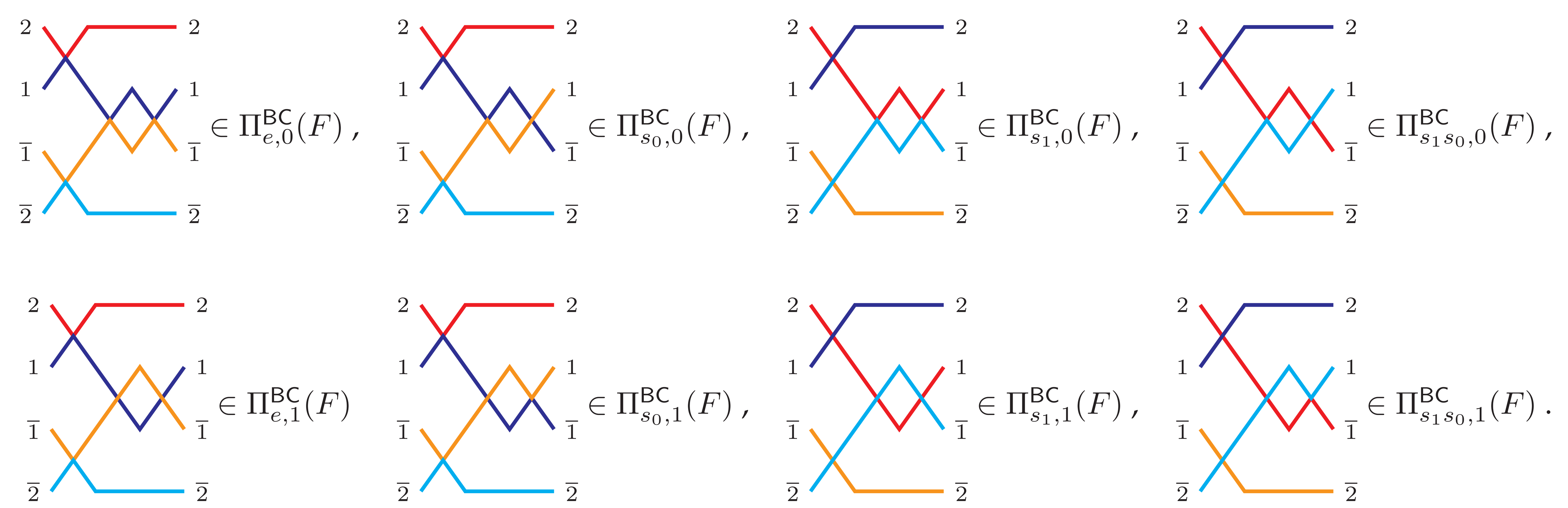
$$\widetilde{C}_{s_{[a_1,b_1]}}(q) \cdots \widetilde{C}_{s_{[a_k,b_k]}}(q) = \sum_{w \in \mathfrak{B}_n} \sum_{d \geq 0} |\Pi_{w,d}^{\mathrm{BC}}(F)| q^d T_w,$$

where $\Pi_{w,d}^{\mathrm{BC}}(F) = \{\pi \in \Pi^{\mathrm{BC}}(F) \,|\, \mathrm{type}(\pi) = w, \mathrm{dfct}^{\mathrm{BC}}(\pi) = d\}$.

EXAMPLE: MULTIPLICATION WITH $(T_e + T_{s_i})$

$F = F_{s_1} \circ F_{s_0} \circ F_{s_0}$ corresponds to $(T_e + T_{s_1})(T_e + T_{s_0})(T_e + T_{s_0})$.

- There are 8 path families in $\Pi^{\mathrm{BC}}(F)$:

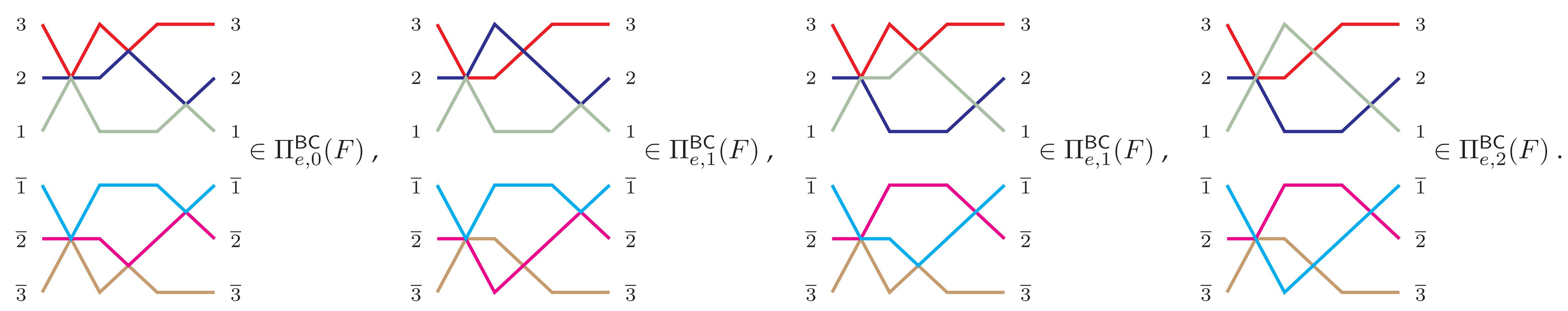


Combining each term gives $(1+q)(T_e + T_{s_0} + T_{s_1} + T_{s_1 s_0})$.

EXAMPLE: MULTIPLICATION WITH $\widetilde{C}_w(q)$

$F = F_{s_{[1,3]}} \circ F_{s_{[2,3]}} \circ F_{s_{[1,2]}}$ corresponds to $\widetilde{C}_{s_{[1,3]}}(q) \widetilde{C}_{s_{[2,3]}}(q) \widetilde{C}_{s_{[1,2]}}(q)$.

- To find the natural coefficient of T_e, observe



Therefore the T_e term has coefficient $(1 + 2q + q^2)$.