Hall–Littlewood–Schubert series and lattice enumeration

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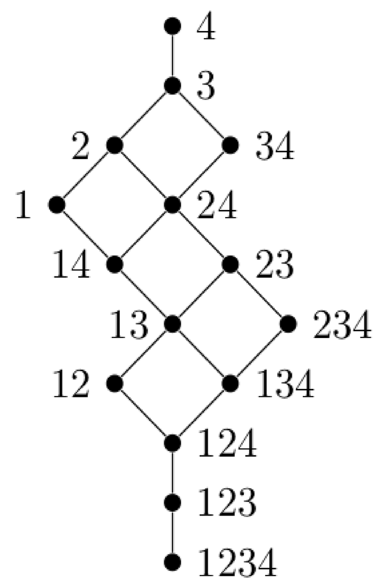
Reduced tableaux and tableaux order

Columns of tableaux are subsets of $[n]$. Their adjacency in a tableaux defines a poset structure T_n on the power set $2^{[n]} \setminus \{\varnothing\}$, called *tableau order*. Here is the Hasse diagram for



$n = 4$

(Reduced) tableaux are chains in this poset, i.e. simplices in the order complex $\Delta(\mathsf{T}_n)$ of T_n.

Fun fact:

HLS_n specializes to the Hilbert series of the Stanley–Reisner ring SR_n of the simplicial complex $\Delta(\mathsf{T}_n)$ of T_n:

$$\mathrm{Hilb}\left(\mathsf{SR}_n, (X_C)_C\right) = \mathsf{HLS}_n\left(0, (X_C)_C\right).$$

Lead question

Given a complete flag

$$\mathbb{Z} \subsetneq \mathbb{Z}^2 \subsetneq \cdots \subsetneq \mathbb{Z}^n,$$

how to enumerate lattices $\Lambda \subsetneq \mathbb{Z}^n$ by the cotypes of all $\Lambda \cap \mathbb{Z}^i$?

Main new object: Hall–Littlewood–Schubert series

For $n \in \mathbb{N}$, define a multivariate rational generating function

$$\mathsf{HLS}_n\left(Y, (X_C)_{\varnothing \neq C \subseteq [n]}\right) = \sum_{T \in \mathrm{rSSYT}_n} \Phi_T(Y) \prod_{C \in T} \frac{X_C}{1 - X_C}.$$

- T ranges over the (reduced, i.e. no repeated columns) *semi-standard Young tableaux* of degree n,
- The *leg polynomial* $\Phi_T(Y) \in \mathbb{Z}[Y]$ is the *leg polynomial* of T, of the general form

$$\Phi_T(Y) = \prod_{i \in I} (1 - Y^{m_i}),$$

for data $(m_i)_{i \in I}$ encoded in the tableau T.

HLS_n grow fast with n–see $n = 3$ below! For more data:

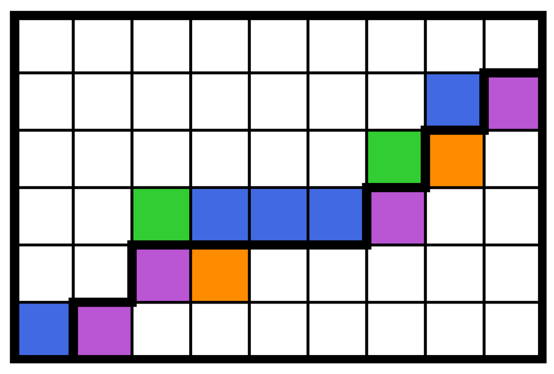
Tableaux and their legs

1	1	1	2	3	8	9
2	3	4	5	7		
3	9	9				
5						
7						

A tableau $T \in \mathrm{rSSYT}_9$ with leg polynomial

$$\Phi_T(Y) = (1 - Y)^4 (1 - Y^2)^3 (1 - Y^3)$$

Tableaux and jigsaws



Duality of tableaux explains duality between various counting problems.

Enjoy the whole story: J. Maglione, C. Voll, *Hall–Littlewood polynomials, affine Schubert series, and lattice enumeration*

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Meta Theorem (MV, 2024)

HLS_n specializes, under judicious substitution of its 2^n variables, to generating functions solving various counting problems. These include:

1. **Affine Schubert series** pertaining to the Lead Problem sketched above, as well as its dual version (replace intersections with projections)
2. **Hecke series** associated with groups of symplectic similitudes (see below)
3. **Hermite–Smith series**, enumerating lattices in $\mathbb{Z}^n$ simultaneously by their Smith and Hermite normal forms (see below)
4. **Zeta functions of integral quiver representations**, enumerating integral subrepresentations of a quiver by their indices

An example: Hall–Littlewood–Schubert series for $n = 3$

$$\begin{aligned}\mathsf{HLS}_3\left(Y, (X_C)_C\right) = \Big(&1 - \mathsf{X}_{1|23} - Y\left(\mathsf{X}_{1|2} + \mathsf{X}_{1|3} + \mathsf{X}_{2|3} + \mathsf{X}_{2|13} + \mathsf{X}_{12|13} + \mathsf{X}_{12|23} + \mathsf{X}_{13|23} + \mathsf{X}_{2|13|23} + \mathsf{X}_{1|2|13|23}\right)\\ &+ Y\left(\mathsf{X}_{1|2|3} + \mathsf{X}_{1|2|13} + \mathsf{X}_{1|2|23} + \mathsf{X}_{1|3|23} + \mathsf{X}_{1|12|23} + \mathsf{X}_{1|13|23} + \mathsf{X}_{12|13|23}\right) + Y^2\left(\mathsf{X}_{1|2|3} + \mathsf{X}_{2|3|13} + \mathsf{X}_{1|3|13} + \mathsf{X}_{2|3|12|13} + \mathsf{X}_{3|12|13} + \mathsf{X}_{3|12|23} + \mathsf{X}_{12|23|13}\right)\\ &- Y^2\left(\mathsf{X}_{3|12} + \mathsf{X}_{1|3|12} + \mathsf{X}_{1|2|3|12} + \mathsf{X}_{1|2|3|13} + \mathsf{X}_{1|3|12|23} + \mathsf{X}_{1|12|13|23} + \mathsf{X}_{2|12|13|23} + \mathsf{X}_{3|12|13|23}\right) - Y^3\left(\mathsf{X}_{2|3|12|13} - \mathsf{X}_{1|2|3|12|13|23}\right)\Big) \Big/ \prod_{\varnothing \neq I \subseteq [3]} (1 - \mathsf{X}_I).\end{aligned}$$

Here we write $\mathsf{X}_{I_1|I_2|\ldots} = X_{I_1} X_{I_2} \cdots$ for subsets $I_1, I_2, \ldots \subset \mathbb{N}$. Spot the numerator's curious palindromicity?! The *self-reciprocity result* below explains this in general.

Self-reciprocity

Theorem (MV)

Hall–Littlewood–Schubert series are highly symmetric:

$$\mathsf{HLS}_n(Y^{-1}, \boldsymbol{X}^{-1}) = (-1)^n Y^{-\binom{n}{2}} X_{[n]} \cdot \mathsf{HLS}_n(Y, \boldsymbol{X}).$$

(Idea of proof: express HLS_n in terms of p-adic integrals.)

Coarse HLS: conjectured nonnegativity

Evaluating $\mathsf{HLS}_n(Y, \boldsymbol{X})$ at $Y = -1$ and $X_C = X$ yields a univariate generating function with—conjecturally–intriguing non-negativity features:

Observation

$$\begin{aligned}\mathsf{HLS}_1(-1, X)(1 - X)^1 &= 1,\\ \mathsf{HLS}_2(-1, X)(1 - X)^3 &= 1 + X^2,\\ \mathsf{HLS}_3(-1, X)(1 - X)^6 &= 1 + X + 6X^2 + 6X^3 + X^4 + X^5,\\ \mathsf{HLS}_4(-1, X)(1 - X)^{10} &= 1 + 5X + 32X^2 + 120X^3 + 226X^4 + \cdots + X^9,\\ \mathsf{HLS}_5(-1, X)(1 - X)^{15} &= 1 + 16X + 179X^2 + 1568X^3 + 8545X^4\\ &\quad + 30448X^5 + 63979X^6 + 83392X^7 + \cdots + X^{14}.\end{aligned}$$

Wanted: an algebro-combinatorial interpretation of these non-negative (?) coefficients. For each n, their sums seem to conicide with the number of maximal paths in the poset of Dyck paths of length $2(n + 1)$.

Symplectic Hecke series

The *symplectic Hecke algebra* $\mathcal{H}_{n,p}$ is the algebra of $\mathbb{C}$-valued functions under convolution on $\mathrm{GSp}_{2n}(\mathbb{Q}_p)$ that are bi-invariant with respect to $\mathrm{GSp}_{2n}(\mathbb{Z}_p)$. Satake gave an isomorphism:

$$\Omega : \mathcal{H}_{n,p} \longrightarrow \mathbb{C}[x_0, x_1, \ldots, x_n].$$

The *Hecke series* is a power series over $\mathcal{H}_{n,p}$ encoding information of many automorphic L-functions. We denote by $\mathsf{H}_{n,p}$ the image of this series under Ω.

Theorem (MV)

Hall–Littlewood–Schubert series "knows" the Hecke series:

$$\mathsf{H}_{n,p}(x_1, \ldots, x_n, x_0 X) = \frac{1}{1 - X} \mathsf{HLS}_n\left(p^{-1}, \left(x_0 X \prod_{i \in C} x_i\right)_C\right).$$

Simultaneous Hermite and Smith

For a lattice $\Lambda \leq \mathbb{Z}_p^n$, the vectors $\lambda(\Lambda), \delta(\Lambda) \in \mathbb{N}_0^n$ record the diagonal entries of the Smith normal form and Hermite normal form, respectively, of Λ. Let $\mu(\Lambda)$ be the consecutive differences of the partition $\lambda(\Lambda)$.

Theorem (MV)

Hall–Littlewood–Schubert series encode both the Hermite and Smith data:

$$\sum_{\Lambda \leq \mathbb{Z}_p^n} \boldsymbol{X}^{\mu(\Lambda)} \boldsymbol{Y}^{\delta(\Lambda)} = \mathsf{HLS}_n\left(p^{-1}, \left(p^{d_n(C)} X_{\#C} \prod_{i \in C} Y_{n+1-i}\right)_C\right).$$

Here, $d_n(C)$ is the dimension of the *Schubert variety* associated with $C \subseteq [n]$.

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