A Combinatorial Model for Affine Demazure Crystals of Levels Zero and One

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Motivation

Crystals are colored, directed graphs that play an important role in the representation theory of quantum groups.

- Important examples include:
  - **highest weight crystals** and their **Demazure subcrystals**,
  - **KR crystals** and their **"Demazure-type" subcrystal** analogues.
- Explicit realizations of crystals are desired, as well as crystal maps that can translate information between different realizations.

KR Crystals and Macdonald Polynomials

Kirillov-Reshetikhin (KR) modules are finite-dimensional $\hat{\mathfrak{g}}$-modules for affine Lie algebras, not of highest weight.

- Most KR modules were shown to admit crystal bases, called **KR crystals**. Throughout we focus on tensor products of single-column KR crystals.
- The **quantum alcove model** uniformly realizes tensor products of single-column KR crystals in untwisted types [1].
- Inside (tensor products of) KR crystals are "Demazure-type" subcrystals, called DARK crystals (**K**irillov-**R**eshetikhin **a**ffine **D**emazure). They correspond to certain quotients of level zero extremal weight modules [2].
- In type A, there is a tabloid model for level one affine Demazure crystals [3].
- **Remark:** The above two crystals were shown to be isomorphic [4].

Nonsymmetric Macdonald polynomials are important families of polynomials with parameters q and t. Specialized to $t = 0$, they are the graded characters of DARK crystals. The grading is by the *energy function*, which is calculated by height statistics from the quantum alcove model [1].

Non-symmetric Quantum Alcove Model

Let $\mathfrak{g}$ be a simple Lie algebra and fix an arbitrary weight μ. There exists a unique maximal-length Weyl group element w such that $\mu = w\lambda$ for a dominant weight λ. The objects of the model depend on two related paths through the weight lattice of $\mathfrak{g}$, depending on μ. These paths can be encoded as sequences of roots:

$$\Gamma = (\gamma_1, \gamma_2, \dots \gamma_m) \qquad \Gamma' = w^{-1}\Gamma = (\beta_1, \beta_2, \dots \beta_m)$$

Definition: A subset $J \subseteq [m]$ is w-*admissible* if there is a path in the quantum Bruhat graph

$$w \leftarrow ws_{\beta_{j_1}} \leftarrow ws_{\beta_{j_1}}s_{\beta_{j_2}} \leftarrow \cdots \leftarrow ws_{\beta_{j_1}}s_{\beta_{j_2}}\dots s_{\beta_{j_s}}.$$

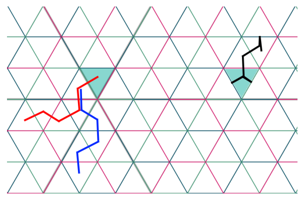
Denote by $\mathcal{A}_w(\Gamma')$ the set of w-admissible subsets.

Remark: We identify a w-admissible subset with a folding of the path Γ along the corresponding hyperplanes.

- We define **crystal operators** e_i, f_i on $\mathcal{A}_w(\Gamma')$.
- The operators act by adding to and (possibly) removing a single element from J.
- In terms of the alcove path, this amounts to folding and (possibly) unfolding Γ.

Example: The alcove paths Γ, Γ' in type A_2 for the weight $\mu = (0, 3, 1)$, and the folded path corresponding to $J = \{1, 2, 5\}$.

$$\Gamma = ((1,2)(1,3)|(1,2)|(3,2)(1,2))$$
$$\Gamma' = ((1,3)(2,3)|(1,3)|(1,2)(1,3))$$



First Theorem

For a weight $\mu = w\lambda$, the w-admissible subsets together with the operators e_i, f_i, for $0 \le i \le n$ defined by

$$e_i(J) = J \setminus \{m\} \cup \{k\}$$
$$f_i(J) = J \setminus \{k\} \cup \{m\},$$

form an abstract $U_q(\hat{\mathfrak{g}})$-crystal isomorphic to the Demazure-type affine crystal $\mathrm{DARK}_w(\lambda)$.

Second Theorem

In type A, there is an explicit crystal isomorphism between w-admissible subsets and semistandard key tabloids, i.e.

- $e_i(\mathrm{fill}(J)) = \mathrm{fill}(e_i(J))$,
- $f_i(\mathrm{fill}(J)) = \mathrm{fill}(f_i(J))$.
- Moreover, the iso preserves weights and translates between the statistics *height* and *major index*.

Tabloid Model for Affine Demazure Crystals in Type A

In type A, the monomial terms of the nonsymmetric Macdonald polynomial at $t = 0$ are indexed by objects called **semistandard key tabloids** (SSKD(μ)), fillings of diagrams of shape μ subject to conditions on pairs and triples of boxes. These come with a level one affine Demazure crystal structure [3].

Example of crystal operator e_4 for $\mu = (0, 0, 4, 0, 6, 2, 2)$:

$$\begin{array}{|c|c|cccc} \hline 2 & 4 \\ \hline 4 & 3 \\ \hline 5 & 5 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|cccc} \hline 2 & 5 \\ \hline 4 & 3 \\ \hline 5 & 4 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 4 \\ \hline \end{array} \qquad\qquad\qquad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 5 \\ \hline \end{array}$$

Relating the Two Models

We construct a map fill : $\mathcal{A}_w(\Gamma') \to SSKD(\mu)$ as well as its inverse.

Starting from the identity, we apply in order the transpositions in Γ corresponding to J:

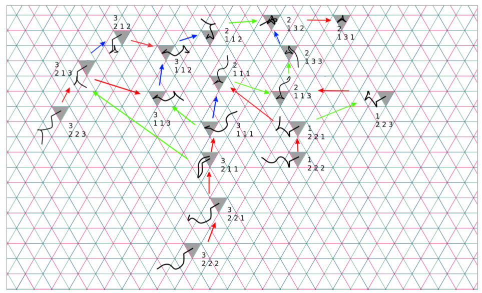
$$\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array} \xrightarrow{(1,2)} \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline 2 \\ \hline \end{array} \xrightarrow{(1,3)} \begin{array}{|c|} \hline \mathbf{2} \\ \hline \mathbf{1} \\ \hline 3 \\ \hline \end{array} \Big| \begin{array}{|c|} \hline 2 \\ \hline \mathbf{1} \\ \hline 3 \\ \hline \end{array} \Big| \xrightarrow{(1,2)} \begin{array}{|c|} \hline 2 \\ \hline \mathbf{3} \\ \hline 1 \\ \hline \end{array}$$

Looking at only the boxes with bold entries gives

$$\mathrm{fill}(J) = \begin{array}{|c|cc} \hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

Given a semistandard key tabloid T, we construct a box-by-box "greedy" algorithm that recovers the w-admissible subset J such that $\mathrm{fill}(J) = T$.

The Crystal $\mathrm{DARK}_w(\lambda)$ for $\lambda = (3, 1, 0)$ and $w = 231$



References

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