What does pattern avoidance have to do with trees and moduli of curves?

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Geometric Background

- $\overline{M}_{0,n+3}$: moduli space of genus 0 stable curves with $n+3$ marked points
- Cohomology classes: $\psi_i := c_1(\mathbb{L}_i)$
- Forgetting map $\pi_i : \overline{M}_{0,i+3} \to \overline{M}_{0,i+2}$: forgets the marked point i
- Pullback ψ_i along forgetting maps: $\omega_i := \pi_n^* \circ \pi_{n-1}^* \circ \cdots \circ \pi_{i+1}^* (\psi_i)$
- $\underline{k} = (k_1, \ldots, k_n)$: composition of n ($k_i \in \mathbb{Z}_{\geqslant 0}$ and $\sum_i k_i = n$)
- Geometric question ([1, 2, 3]): embed $\overline{M}_{0,n+3}$ into product of projective spaces, study map using ω-classes.

Theorem 1 ([2]). $\int_{\overline{M}_{0,n+3}} \omega^{\underline{k}} = \left\langle {n \atop \underline{k}} \right\rangle = \left|\mathrm{Slide}^{\omega}(\underline{k})\right| = \left|\mathrm{Tour}(\underline{k})\right|$.

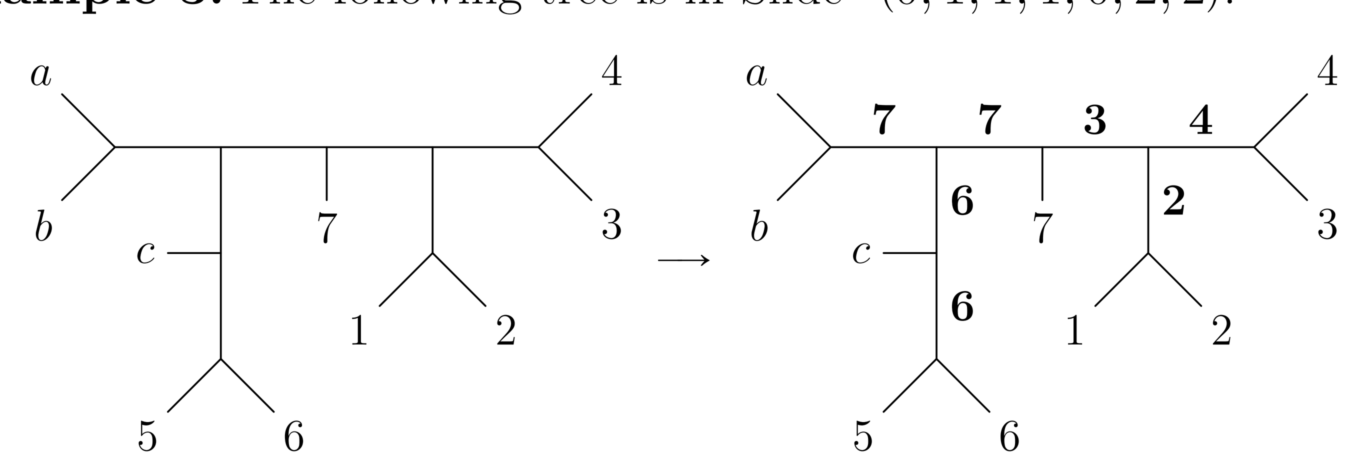
- $\left\langle {n \atop \underline{k}} \right\rangle$: **asymmetric multinomial coefficients** defined by $\left\langle {1 \atop 1} \right\rangle = 1$ and
$$\left\langle {n \atop \underline{k}} \right\rangle = \sum_{j=i+1}^{n} \left\langle {n-1 \atop \underline{k}^{(j)}} \right\rangle.$$
- $\mathrm{Tour}(\underline{k})$, $\mathrm{Slide}^{\omega}(\underline{k})$: sets of trivalent trees
- **GOAL**: Find bijection $\mathrm{Tour}(\underline{k}) \leftrightarrow \mathrm{Slide}^{\omega}(\underline{k})$

Slide Trees

Definition 2 (Slide labeling algorithm). A tree T is in $\mathrm{Slide}^{\omega}(\underline{k})$ (resp. $\mathrm{Slide}^{\psi}(\underline{k})$) if the following algorithm finishes successfully:

0. Start with $\ell = n$.
1. **Choose next edge to label:** Let e be the first unlabeled internal edge on path from leaf ℓ to a. (If none exist, then labeling fails.)
2. **Verify that label is valid:** Let m_1 be smallest leaf label on the same side of e as ℓ, and m_2 the smallest on the same side of e as a, excluding the branch containing a itself. If $\ell \geqslant m_1 \geqslant m_2$, (resp. $m_1 \geqslant m_2$,) then label e with ℓ. Else, terminate.
3. **Iterate:** If ℓ has labeled k_ℓ edges, decrement ℓ. If $\ell = 0$, we're done.
4. **Contract labeled edges.**

Example 3. The following tree is in $\mathrm{Slide}^{\omega}(0, 1, 1, 1, 0, 2, 2)$.



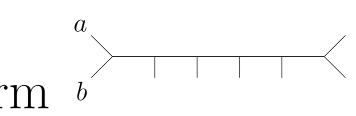
Patterns and Caterpillars

- **Barred patterns:** Let some entries of pattern π be barred. For word τ to contain π, it must have a subword with the relative order of the non-barred portion of π that is **not** a subword of all of π.

Example 7. $\tau = 1\underline{2}34\underline{5}6$ contains $\pi = 23\overline{1}$, while $\tau = 234561$ avoids π.

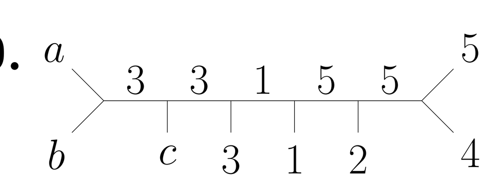
- **Vincular patterns:** Impose adjacency conditions, use a dash to indicate entries of π that need not be adjacent to each other in τ.

Example 8. $\tau = 3\underline{25}4\underline{1}$ contains $\pi = 23\text{–}1$, while $\tau = 43152$ avoids π.

- A tree of the form  is called a caterpillar tree. Denote the set of caterpillars by $\mathrm{Cat}^{\omega}(\underline{k}) \subseteq \mathrm{Slide}^{\omega}(\underline{k})$.

Theorem 9 (R.-B.). *Let $\underline{k}$ be a reverse-Catalan composition, and let w be a word of composition $\underline{k}$. Then:*

- $\mathrm{tree}(w) \in \mathrm{Cat}^{\psi}(\underline{k})$ *if and only if* $w \in \mathrm{Av}_{\underline{k}}\left(2\text{–}1\text{–}2, 23\text{–}\overline{2}\text{–}1\right)$ *and* $\mathrm{TotalRep}_w(i) + \ell_i \geqslant \mathrm{z}(i)$ *for all i, and*
- $\mathrm{tree}(w) \in \mathrm{Cat}^{\omega}(\underline{k})$ *if and only if* $w \in \mathrm{Av}_{\underline{k}}\left(2\text{–}1\text{–}2, 23\text{–}\overline{2}\text{–}1\right)$ *and* $\mathrm{BigRep}_w(i) \geqslant \mathrm{z}(i)$ *for all i.*

Example 10.  is in $\mathrm{Slide}^{\psi}(1, 0, 2, 0, 2)$, but not $\mathrm{Slide}^{\omega}(1, 0, 2, 0, 2)$.

The case $\underline{k} = (1, 1, \ldots, 1)$

Theorem 4 ([2]). *There exists a bijection*
$$\phi : \mathrm{Av}_n(23\text{–}1) \longleftrightarrow \mathrm{Cat}(1, 1, \ldots, 1).$$

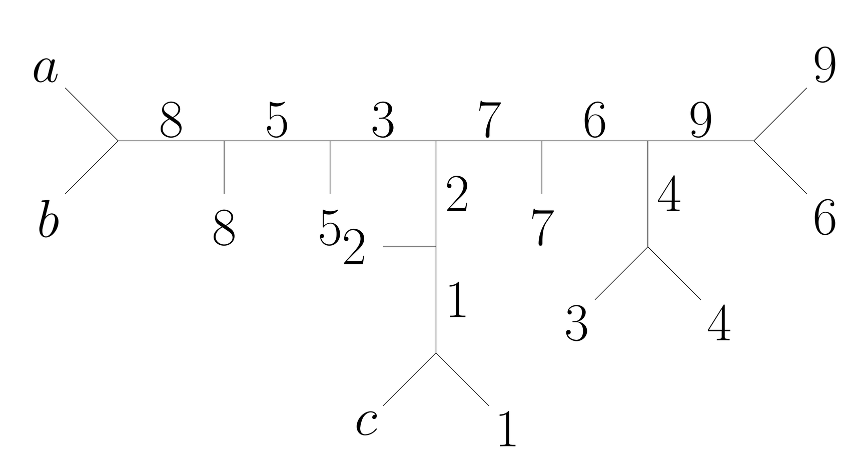
- Given permutation τ, let x, y, z be *earliest* 23–1 pattern in τ, and write $\tau = wxyuzv$

Theorem 5 (R.-B.). *Define a map ρ recursively as follows.*

- *If $\tau \in \mathrm{Av}_n(23\text{–}1)$, then $\rho(\tau) := \phi(\tau)$.*
- *Otherwise, let $\rho(\tau)$ be tree formed by "splicing" $\rho(wxyu)$ and $\rho(wxzv)$ together.*

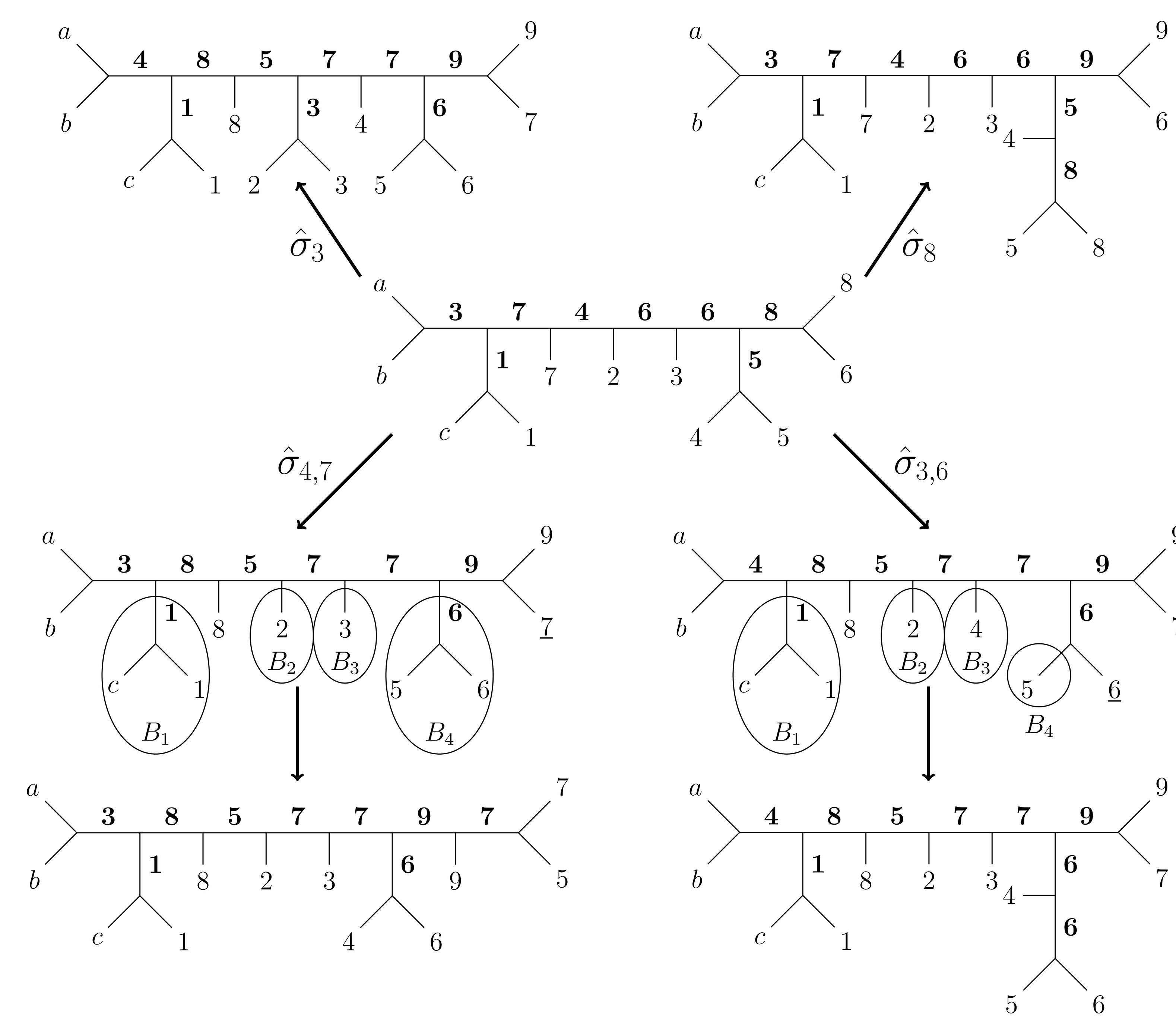
The resulting map $\rho : \mathfrak{S}_n \longleftrightarrow \mathrm{Slide}(\underline{k})$ is a bijection.

Example 6. Let $\tau = 853\underline{7}6942\underline{1}$. Then, $\rho(\tau)$ is as follows:



Main Bijection (R.-B.)

- We show that $\mathrm{Slide}^{\omega}(\underline{k})$ satisfy *asymmetric multinomial* recurrence.
- **Idea**: Build bijection
$$\mathrm{Slide}^{\omega}(\underline{k}) \longleftrightarrow \bigsqcup_{j=i+1}^{n} \mathrm{Slide}^{\omega}\left(\underline{k}^{j}\right)$$
- Define maps $\hat{\sigma}_{i,j}, \hat{\sigma}_j$ from $\mathrm{Slide}^{\omega}\left(\underline{k}_j'\right)$ to $\mathrm{Slide}^{\omega}(\underline{k})$, for certain compositions $\underline{k}_j'$ of $n-1$
- Piece these together to form $\Sigma_{\underline{k}} : \bigsqcup_{j=i+1}^{n} \mathrm{Slide}^{\omega}\left(\underline{k}^{j}\right) \to \mathrm{Slide}^{\omega}(\underline{k})$
- For bijection between $\mathrm{Tour}(\underline{k})$ and $\mathrm{Slide}^{\omega}(\underline{k})$, unwind both recurrences iteratively
- Examples of $\hat{\sigma}_{i,j}$ and $\hat{\sigma}_j$:



References

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[2] Maria Gillespie, Sean Griffin, and Jake Levinson. Degenerations and multiplicity-free formulas for products of ψ and ω classes on $\overline{M}_{0,n}$. *Mathematische Zeitschrift*, 304, 07 2023.

[3] Mikhail M Kapranov. Veronese curves and Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$. *J. Algebraic Geom*, 2:239–262, 1993.

[4] Andrew Reimer-Berg. Insertion algorithms and pattern avoidance on trees arising in the Kapranov embedding of $\overline{M}_{0,n+3}$, 2025. Preprint, http://arxiv.org/abs/2504.17098.