A hidden symmetry of refined canonical stable Grothendieck polynomials

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Grothendieck Polynomials

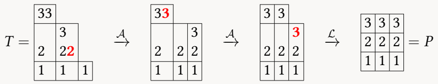
Grothendieck by Lascoux and Schützenberger (1982), introduced for studying the Grothendieck ring of vector bundles on a flag variety

+ **stable** by Fomin and Kirillov (1994)
+ **canonical** by Yeliussizov (2017)
+ **refined** by Hwang–Jang–Kim–Song–Song (2024)

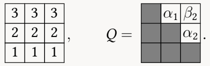
$$G_\lambda(\boldsymbol{x}_n;\boldsymbol{\alpha},\boldsymbol{\beta}) := \frac{\det\left(x_j^{\lambda_i+n-i}\prod_{\ell=1}^{i-1}(1-\beta_\ell x_j)\prod_{\ell=1}^{\lambda_i}(1-\alpha_\ell x_j)^{-1}\right)_{1\le i,j\le n}}{\prod_{1\le i<j\le n}(x_i-x_j)}$$

Uncrowding I [Pan–Pappe–Poh–Schilling, 2022]

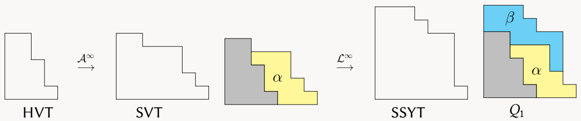
$\mathcal{A}$: single-arm-uncrowding map, and $\mathcal{L}$: single-leg-uncrowding map



This shows that $\mathcal{U}_{\mathcal{LAA}}(T) = (P, Q)$, where



Question 1



(Partial) description: Q_1 is an (α,β)-sorted, α-column-strict, β-row-strict flagged-mixed tableau.
Question1. Can we describe the image of the uncrowding map?

Two Combinatorial Models [Hwang–Jang–Kim–Song–Song, 2024]

Hook-valued tableaux $\mathrm{HVT}(\lambda)$

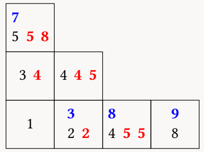
$$G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{T\in \mathrm{HVT}(\lambda)} \mathrm{wt}(T),$$

where $\mathrm{wt}(T) = \prod_{i\ge 1}\alpha_i^{(\#\text{ of arm entries in column } i)}\beta_i^{(\#\text{ of leg entries in row } i)}x_i^{(\#\text{ of } i\text{'s in } T)}$.

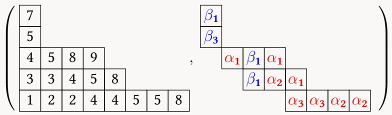
Semistandard Young tableaux, and exquisite tableaux $\mathrm{SSYT}(\mu)\times\mathrm{EXQ}(\mu/\lambda)$

$$G_\lambda(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta}) = \sum_{\mu\supseteq\lambda}\sum_{(P,Q)\in\mathrm{SSYT}(\mu)\times\mathrm{EXQ}(\mu/\lambda)} \mathrm{wt}(Q)\boldsymbol{x}^P,$$

where $\mathrm{wt}(Q)$ is the product of its entries.

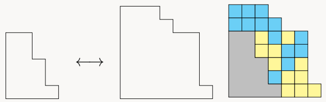


$\mathrm{wt} = \alpha_1^3\alpha_2^3\alpha_3^2\beta_1^3\beta_3x_1x_2^2x_3^2x_4^4x_5^5x_7x_8^3x_9$



$\mathrm{wt} = x_1x_2^2x_3^2x_4^4x_5^5x_7x_8^3x_9\alpha_1^3\alpha_2^3\alpha_3^2\beta_1^3\beta_3$

Question 2



(Full) description: Q_2 is a flagged-mixed tableau, and $c_\beta^+(Q_2)$ is totally column-strict.
Question2. Can we find a combinatorial proof of the equivalence of the two models?

Notation: Flagged-mixed and Exquisite Tableaux

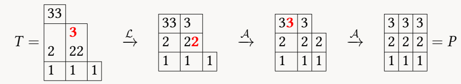
A *flagged-mixed tableau* is a filling T of the cells with elements in $\{\alpha_k \mid k\in\mathbb{Z}_{>0}\}\cup\{\beta_k\mid k\in\mathbb{Z}\}$ satisfying:

1. If $T(i,j)=\alpha_k$, then $0<k<j$.
2. If $T(i,j)=\beta_k$, then $0<k<i$.

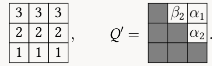
The *content* of (i,j) is $c(i,j) := j-i$. The tableau $c_\beta^+(T)$ is obtained from T by $\beta_r\mapsto\beta_{r+c(i,j)}$.

An *exquisite tableau* is a flagged-mixed tableau T such that $c_\beta^+(T)$ is totally column-strict.

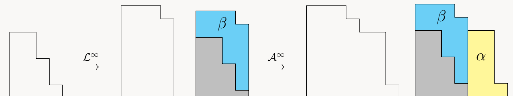
Uncrowding II [Pan–Pappe–Poh–Schilling, 2022]



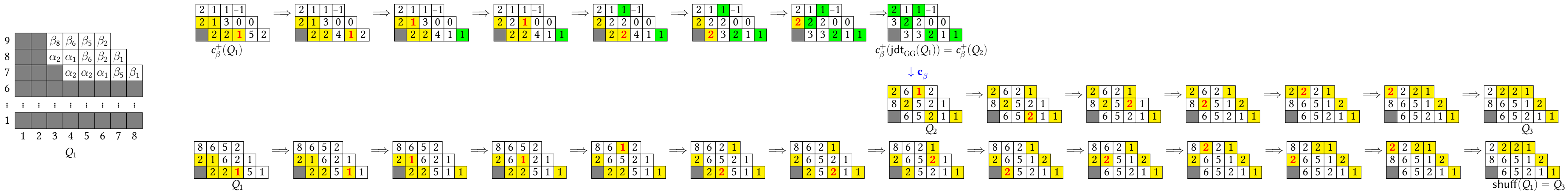
This shows that $\mathcal{U}_{\mathcal{AAL}}(T) = (P, Q')$, where



Question 3

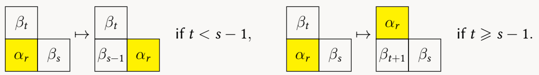


(Partial) description: Q_3 is a (β,α)-sorted, α-column-strict, β-row-strict flagged-mixed tableau.
Question3. What is the relation between Q_1 and Q_3?



Goulden–Greene's jeu de taquin jdt_{GG} on $c_\beta^+(Q)$

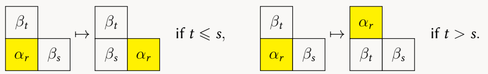
1. Find the smallest r and choose the rightmost α_r that is out of order in Q.
2. Apply the GG-jdt slide at α_r.



3. Repeat (1)-(2) until no entries are out of order.

(Jeu de taquin) Shuffle $\mathrm{shuff}(Q)$

1. Find the smallest r and choose the rightmost α_r.
2. Apply the usual 'jeu de taquin' slide at α_r.



3. Repeat (1)-(2) until possible.

Answers to Questions [Jang–Kim–Pan–Pappe–Schilling, 2025+]

A tableau T is called a *biflagged tableau* (BFT) if T is α-column-strict, β-row-strict, and (α,β)-sorted with the additional condition that both T and $\mathrm{shuff}(T)$ are flagged-mixed tableaux.

Answer 1. $Q_1\in\mathsf{BFT}$.

Answer 2. The following map is a bijection:

$$\mathrm{HVT}(\lambda)\xrightarrow{\mathcal{A}^\infty\mathcal{L}^\infty}\bigcup_{\mu\supseteq\lambda}(\mathrm{SSYT}(\mu)\times\mathsf{BFT}(\mu/\lambda))\xrightarrow{\mathrm{id}\times\mathrm{jdt}_{GG}}\bigcup_{\mu\supseteq\lambda}(\mathrm{SSYT}(\mu)\times\mathsf{EXQ}(\mu/\lambda)).$$

Answer 3. $Q_3 = \mathrm{shuff}(Q_1)$.

Our main idea uses two key ingredients:

- By [Krattenthaler, 1996], the map jdt_{GG} is a bijection
  $\{\alpha\text{-column-strict, }\beta\text{-row-strict, and }(\alpha,\beta)\text{-sorted tableaux}\}\to\{E : c_\beta^+(E)\text{ is a totally column-strict}\}$.
- By [Benkart–Sottile–Stroomer, 1996], the maps jdt_{GG} and shuff are special cases of (partial) tableau switching.

References

1. Benkart, Sottile, Stroomer. *Tableau switching: algorithms and applications*, J. Combin. Theory Ser. A, 1996.
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