A charge monomial basis of the Garsia–Procesi ring

Mitsuki Hanada[1]

[1]Department of Mathematics, UC Berkeley

Combinatorial Definitions

- A **standard Young tableau** (SYT) of shape $\lambda \vdash n$ is a filling of λ where $\{1,\dots,n\}$ appear exactly once and the entries are increasing within the rows and columns.

$$\lambda = (4,2,1) \qquad \begin{array}{|c|c|c|c|}\hline 6 \\ \cline{1-1} 2 & 5 \\ \cline{1-2} 1 & 3 & 4 & 7 \\ \hline\end{array}.$$

- For a permutation $\sigma \in S_n$, the **Major index** statistic is $\mathrm{maj}(\sigma) := \sum_{i:\sigma_i>\sigma_{i+1}} i$.
- The **Robinson-Schensted correspondence** gives a bijection

$$S_n \Leftrightarrow \{(P,Q) \mid P,Q \in \mathrm{SYT}_n, \mathrm{shape}(P) = \mathrm{shape}(Q)\}$$

where $\sigma \mapsto (P(\sigma), Q(\sigma))$.

Charge

The **Charge** statistic on permutations is related to maj by $\mathrm{maj}(\sigma) = \mathrm{charge}(w)$ for $\sigma = \mathrm{rev}(w^{-1})$.

The **charge word** $\mathrm{c}(w)$ of w is a labeling of w given by:

1. label 1 by 0,
2. if we label i by k, we label $i+1$ by

$$\begin{cases} k+1 & \text{if } i+1 \text{ is to the right of } i \\ k & \text{if } i \text{ is to the left of } i \end{cases}$$

The word consisting of these labels is the charge word $\mathrm{c}(w)$.

$$w = 4\ 2\ 1\ 5\ 3$$
$$\mathrm{c}(w) = 1\ 0\ 0\ 2\ 1$$

Here are some facts about charge:

- $\mathrm{charge}(w)$ is the sum of the entries in $\mathrm{c}(w)$.
- For SYT T, define $\mathrm{charge}(T) := \mathrm{charge}(\mathrm{rw}(T))$ where $\mathrm{rw}(T)$ is the **row reading word** of T.
- If $P(w) = P(w')$, then $\mathrm{charge}(w) = \mathrm{charge}(w')$.

Frobenius character

For a S_n representation V, the **Frobenius character** of V ($\mathrm{Frob}(V)$) is a symmetric function encoding its decomposition into irreducibles via the map $\mathrm{ch}(V_\lambda) \mapsto s_\lambda[X]$.

$$\mathrm{Frob}(V_{(2,1)} \oplus V_{(1,1,1)}) = s_{(2,1)} + s_{(1,1,1)}.$$

For graded $V = \oplus_{d\geq 0} V_d$, the **graded Frobenius character** $\mathrm{Frob}_q(V)$ is $\mathrm{Frob}_q(V) = \sum_{d\geq 0} q^d\,\mathrm{Frob}(V_d)$.

The coinvariant ring

The **coinvariant ring** R_{1^n} is defined to be

$$R_{1^n} = \mathbb{C}[\mathbf{x}]/\langle e_k(\mathbf{x}) \text{ for } k \in \{1,\cdots,n\}\rangle$$

where $e_k(\mathbf{x}) = \sum_{i_1<i_2<\cdots<i_k} x_{i_1}x_{i_2}\cdots x_{i_k}$.

As S_n-representations, R_{1^n} is isomorphic to...

- Regular rep of S_n(ungraded)
- Cohomology ring of the flag variety (as graded S_n reps)

Monomial bases of coinvariant ring

Remark

- $\mathrm{Hilb}_q(R_{1^n}) = [n]_q!$ where $[k]_q = (1+q+\cdots q^{k-1})$.
- $\mathrm{Frob}_q(R_{1^n}) = \sum_{T\in\mathrm{SYT}_n} q^{\mathrm{charge}(T)} s_{\mathrm{shape}(T^t)}$

There are two well-known monomial base of R_{1^n}, both indexed by permutations $\sigma \in S_n$

- Artin basis

$$\{f_\sigma(\mathbf{x}) = \prod_{i<j,\sigma_i>\sigma_j} x_{\sigma_i}\}$$

where $\deg(f_\sigma) = \mathrm{inv}(\sigma)$.

- Descent basis

$$\{g_\sigma(\mathbf{x}) = \prod_{i,\sigma_i>\sigma_{i+1}} x_{\sigma_1}\cdots x_{\sigma_i}\}$$

where $\deg(g_\sigma) = \mathrm{maj}(\sigma)$.

Why are these bases nice?

Both bases are compatible with the Hilbert series of R_{1^n}:

$$\sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma)} = [n]_q! = \mathrm{Hilb}_q(R_{1^n}).$$

Garsia–Procesi rings

For $\mu \vdash n$, the **Garsia-Procesi ring** R_μ are quotients of the coinvariant ring, defined to be:

$$R_\mu = \mathbb{C}[\mathbf{x}]/I_\mu$$

where the ideal I_μ is generated by

$$\{e_d(S) \mid S \subset \{x_1,\dots,x_n\}, |S| - p^n_{|S|}(\mu) < d \leq |S|\}$$

where $p^n_k(\mu)$ is the number of boxes that are **not** in the first $(n-k)$ columns of the Young diagram of μ.

As S_n-representations, R_μ is isomorphic to...

- the induction of the trivial $1\uparrow^{S_n}_{S_\mu}$ (ungraded)
- Cohomology ring of Springer fibers indexed by μ (Springer fiber $\subset$ Flag variety)

Modified Hall-Littlewood polynomials

For $\mu \vdash n$, we have

$$\mathrm{Frob}_q(R_\mu) = \tilde{H}_\mu[X;q]$$

where $\tilde{H}_\mu[X;q]$ is the *modified Hall-Littlewood polynomial.*

Theorem (Lascoux 1989)

$$\tilde{H}_\mu[X;q] = \sum_{\substack{T\in\mathrm{SYT}_n \\ \mathrm{ctype}(T^t)\trianglerighteq\mu}} q^{\mathrm{charge}(T)} s_{\mathrm{shape}(T^t)},$$

where T^t denotes the transpose of T.

The **catabolizability type** $\mathrm{ctype}(T)$ of a SYT is a partition. There exists a cocharge/shape preserving bijection:

$\{T \in \mathrm{SYT}_n \mid \mathrm{ctype}(T)\trianglerighteq\mu\} \leftrightarrow \{S \in \mathrm{SSYT} \text{ with weight } \mu\}$.

Question

Find a description of a monomial basis of R_μ that is

- a subset of the Artin basis/Descent basis,
- compatible with

$$\mathrm{Frob}_q(R_\mu) = \sum_{\substack{T\in\mathrm{SYT}_n \\ \mathrm{ctype}(T^t)\trianglerighteq\mu}} q^{\mathrm{charge}(T)} s_{\mathrm{shape}(T^t)}$$

$$\mathrm{Hilb}_q(R_\mu) = \sum_{\substack{w\in S_n \\ \mathrm{ctype}(P(w)^t)\trianglerighteq\mu}} q^{\mathrm{charge}(w)}.$$

.

Theorem

The set

$$\{\mathbf{x}^{\mathrm{c}(w)} \mid w \in S_n,\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu\}$$

is a monomial basis of R_μ.

There is an alternative construction of this basis due to Carlsson–Chou [1] but it is not compatible with $\mathrm{Hilb}_q(R_\mu)$.

Example: basis of R_μ for $\mu = (2,1,1)$

S	$\{w \mid P(w) = S\}$	$\{x^{\mathrm{c}(w)} \mid P(w) = S\}$
$\begin{array}{ccc}2\\1&3&4\end{array}$	$\{2134,\ 2314,\ 2341\}$	$\{x_3x_4^2,\ x_2x_4^2,\ x_2x_3^2\}$
$\begin{array}{cc}2&4\\1&3\end{array}$	$\{2143\ ,\ 2413\}$	$\{x_3x_4,\ x_2x_4\}$
$\begin{array}{cc}4\\2\\1&3\end{array}$	$\{4213,\ 4231,\ 2431\}$	$\{x_1x_4,\ x_1x_3,\ x_2x_3\}$
$\begin{array}{cc}3\\2\\1&4\end{array}$	$\{3214,\ 3241\ ,\ 3421\}$	$\{x_4,\ x_3,\ x_2\}$
$\begin{array}{c}4\\3\\2\\1\end{array}$	$\{4321\}$	$\{1\}$

Why is this basis nice?

- It is a subset of the descent basis of R_{1^n}. (charge $\leftrightarrow$ maj)
- It is compatible with

$$\mathrm{Hilb}_q(R_\mu) = \sum_{\substack{w\in S_n \\ \mathrm{ctype}(P(w)^t)\trianglerighteq\mu}} q^{\mathrm{charge}(w)}.$$

- It gives an elementary proof of $\mathrm{Frob}_q(R_\mu) = \tilde{H}_\mu[X;q]$ that only depends on the **ungraded** Frobenius character.

This gives the **first direct connection** between the structure of R_μ as a ring and the combinatorial formula for $\tilde{H}_\mu[X;q]$.

$\mathrm{Frob}_q(R_\mu) = \tilde{H}_\mu[X;q]$

For $\mathbb{C}S_n$-module V, $\mathrm{Frob}_q(V)$ is determined by

$$\mathrm{Hilb}_q(N_\gamma V) = \langle e_\gamma, \mathrm{Frob}_q(V)\rangle$$

for all $\gamma \vdash n$, where $N_\gamma = \sum_{\sigma\in S_\gamma} \mathrm{sgn}(\sigma)\sigma$.

We also know

$$\langle e_\gamma,\ \tilde{H}_\mu[X;q]\rangle = \sum_{\substack{w\in S_n \\ \mathrm{ctype}(P(w)^t)\trianglerighteq\mu \\ \mathrm{des}(w)\subset\{\gamma_1,\gamma_1+\gamma_2,\dots,\gamma_1+\cdots+\gamma_{l-1}\}}} q^{\mathrm{charge}(w)}.$$

Proposition

Let $\mu,\gamma \vdash n$. The set

$$\{N_\gamma \mathbf{x}^{\mathrm{c}(w)} \mid w \in S_n,\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu,\ \mathrm{des}(w) \subset \{\gamma_1, \gamma_1+\gamma_2, \dots, \gamma_1+\cdots+\gamma_{l-1}\}\}.$$

is a basis of $N_\gamma R_\mu$ where $\mathrm{des}(w) = \{i \mid w_i > w_{i+1}\}$.

This implies the following:

Corollary

We have

$$\mathrm{Frob}_q(R_\mu) = \tilde{H}_\mu[X;q].$$

Proof.

$$\mathrm{Hilb}_q(N_\gamma R_\mu) = \sum_{\substack{w\in S_n \\ \mathrm{ctype}(P(w)^t)\trianglerighteq\mu \\ \mathrm{des}(w)\subset\{\gamma_1,\gamma_1+\gamma_2,\dots,\gamma_1+\cdots+\gamma_{l-1}\}}} q^{\mathrm{charge}(w)}. \quad \square$$

Example: basis of $N_\gamma R_\mu$ for $\gamma = (2,2)$

There are 5 SYT P that satisfy $\mathrm{ctype}(P^t) \trianglerighteq (2,1,1)$:

$$\begin{array}{ccc}2\\1&3&4\end{array},\ \begin{array}{cc}2&4\\1&3\end{array},\ \begin{array}{cc}4\\2\\1&3\end{array},\ \begin{array}{cc}3\\2\\1&4\end{array},\ \begin{array}{c}4\\3\\2\\1\end{array}.$$

Note that $\mathrm{des}(w) = \mathrm{des}(Q(w))$. There are 3 SYT Q such that $\mathrm{des}(Q) \subset \{2\} = \{\gamma_1\}$:

$$\begin{array}{cccc}1&2&3&4\end{array},\ \begin{array}{ccc}3\\1&2&4\end{array},\ \begin{array}{cc}3&4\\1&2\end{array}.$$

We have two pairs (P,Q) where P,Q are the same shape

$$\left(\begin{array}{ccc}2\\1&3&4\end{array}, \begin{array}{ccc}3\\1&2&4\end{array}\right) \leftrightarrow\ w = 2314 \leftrightarrow\ \mathrm{c}(w) = 0102,$$

$$\left(\begin{array}{cc}2&4\\1&3\end{array}, \begin{array}{cc}3&4\\1&2\end{array}\right) \leftrightarrow\ w = 2413 \leftrightarrow\ \mathrm{c}(w) = 0101.$$

The basis of $N_\gamma R_\mu$ is $\{N_\gamma(x_2x_4^2), N_\gamma(x_2x_4)\}$.

References

1. E.Carlsson and R. Chou, *A descent basis for the Garsia-Procesi module*, Adv. Math 457 (2024)
2. A.M. Garsia and C.Procesi, *On certain graded Sn-modules and the q-Kostka polynomials*, Adv. Math 94.1 (1992)
3. M.Hanada, *A charge monomial of the Garsia-Procesi rings*, arXiv: 2410.15514
4. A.Lascoux, *Cyclic permutations on words, tableaux and harmonic polynomials*, Proc. of the Hyderabad Conference on Algebraic Groups (1989)