Filtered RSK and matrix Schubert varieties

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Introduction

Let $\mathbf{GL} := GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$. $\mathbf{GL}$ acts on the coordinate ring of complex $m \times n$ matrices $\mathbb{C}[\mathrm{Mat}_{m,n}]$ by:

$$(g,h)\cdot f(M) = f(gMh^T),\ (g,h) \in \mathbf{GL},\ M \in \mathrm{Mat}_{m,n}.$$

Since $\mathbf{GL}$ fixes graded components of $\mathbb{C}[\mathrm{Mat}_{m,n}]$, every graded component of $\mathbb{C}[\mathrm{Mat}_{m,n}]$ decomposes into a direct sum of external tensor products $V_\lambda(m) \boxtimes V_\mu(n)$ where $V_\lambda(m)$ is the Weyl module for GL_m corresponding to the partition λ. It can be shown that

$$\mathbb{C}[\mathrm{Mat}_{m,n}] \simeq_{\mathbf{GL}} \bigoplus_{\lambda} V_\lambda(m) \boxtimes V_\lambda(n),$$

where the sum runs over all partitions λ with length at most $\min(m,n)$. This is the classical Cauchy identity.

Any Levi subgroup $\mathbf{L}_{\mathbf{I}|\mathbf{J}} := \mathbf{L}_{\mathbf{I}} \times \mathbf{L}_{\mathbf{J}} \leq \mathbf{GL}$, where

$$\mathbf{L}_{\mathbf{I}} = GL_{i_1 - i_0} \times \ldots \times GL_{i_k - i_{k-1}},\ \ \mathbf{I} = \{0 = i_0 < i_1 < \ldots < i_k = m\},$$

also acts on $\mathbb{C}[\mathrm{Mat}_{m,n}]$ by restriction. As an $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-representation, $\mathbb{C}[\mathrm{Mat}_{m,n}]$ decomposes into a direct sum of $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-irreducibles:

$$\mathbb{C}[\mathrm{Mat}_{m,n}] \simeq_{\mathbf{L}_{\mathbf{I}|\mathbf{J}}} \bigoplus_{\underline{\lambda}|\underline{\mu}} (V_{\underline{\lambda}}(m) \boxtimes V_{\underline{\mu}}(n))^{c_{\underline{\lambda}|\underline{\mu}}}$$

Here, $\underline{\lambda}, \underline{\mu}$ are tuples of partitions. The multiplicities $c_{\underline{\lambda}|\underline{\mu}}$ may be expressed in terms of the Littlewood-Richardson coefficients.

Let $I \subseteq \mathbb{C}[\mathrm{Mat}_{m,n}]$ be an ideal stable under the action of some $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$. Then

$$\mathbb{C}[\mathrm{Mat}_{m,n}]/I \simeq_{\mathbf{L}_{\mathbf{I}|\mathbf{J}}} \bigoplus_{\underline{\lambda}|\underline{\mu}} (V_{\underline{\lambda}}(m) \boxtimes V_{\underline{\mu}}(n))^{c^I_{\underline{\lambda}|\underline{\mu}}}.$$

Main Question. *What is a combinatorial rule for computing the multiplicities $c^I_{\underline{\lambda}|\underline{\mu}}$?*

Crystals and filteredRSK

Basis vectors for a Weyl module $V_\lambda(m)$ are indexed by semistandard Young tableaux of shape λ with content $[m]$. The set of all semistandard Young tableaux of shape λ may be arranged into a *crystal graph* $\mathfrak{B}_\lambda(m)$ using Kashiwara's *crystal operators*. Each $\mathfrak{B}_\lambda(m)$ has a unique source, the *highest weight tableau* of shape λ.

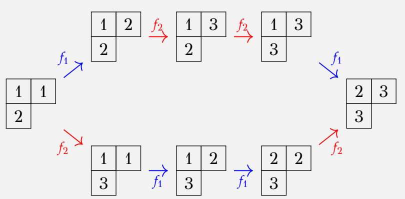

Figure 1: $\mathbf{GL}$*-Crystal for* $V_{(2,1)}(3)$

The crystal graph for a direct sum $V_\lambda(m) \oplus V_\mu(m)$ of irreducible representations is the disjoint union of the crystals $\mathfrak{B}_\lambda(m)$ and $\mathfrak{B}_\mu(m)$. The crystal graph for a $GL_m \times GL_n$ representation $V_\lambda(m) \boxtimes V_\mu(n)$ is the Cartesian product $\mathfrak{B}_\lambda(m) \square \mathfrak{B}_\mu(n)$ of the graphs $\mathfrak{B}_\lambda(m), \mathfrak{B}_\mu(n)$. We call $\mathfrak{B}_\lambda(m) \square \mathfrak{B}_\mu(n)$ a *bicrystal.*

Crystal graphs behave nicely under Levi branching. Given a Weyl module $V_\lambda(m)$ and a Levi subgroup $\mathbf{L}_{\mathbf{I}} \leq GL_m$, the crystal graph $\mathfrak{B}^{\mathbf{L}_{\mathbf{I}}}_\lambda(m)$ for $V_\lambda(m)$ as an $\mathbf{L}_{\mathbf{I}}$ representation is the crystal formed from $\mathfrak{B}_\lambda(m)$ by removing all arrows labeled with an f_i for $i \in \mathbf{I}$.

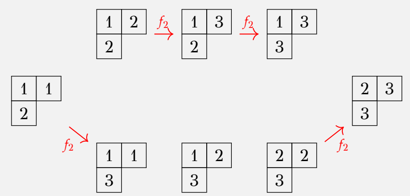

Figure 2: *Branching* $V_{(2,1)}(3)$ *for* $\mathbf{L}_{\{0,1,3\}}$

Danilov-Koshevoi [1] and van Leeuwen [3] give the set $\mathfrak{B}$ of all monomials in $\mathbb{C}[\mathrm{Mat}_{m,n}]$ a bicrystal structure by "pulling back" bicrystal operators on SSYT through RSK.

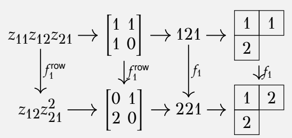

Figure 3: *Pulling back tableaux operators to monomials*

This bicrystal structure yields a manifestly positive combinatorial rule for the multiplicity of $V_{\underline{\lambda}} \boxtimes V_{\underline{\mu}}$ in the decomposition of $\mathbb{C}[\mathrm{Mat}_{m,n}]$ as an $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-representation; namely, $c_{\underline{\lambda}|\underline{\mu}}$ is the number of highest weight matrices indexing a connected component of the $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-crystal $\mathfrak{B}^{\mathbf{L}_{\mathbf{I}|\mathbf{J}}}$ that is isomorphic to the crystal for $V_{\underline{\lambda}} \boxtimes V_{\underline{\mu}}$. The highest weight matrices are those which correspond to tuples of highest weight tableaux $(T_{\underline{\lambda}}|T_{\underline{\mu}})$ via *filtered RSK.*

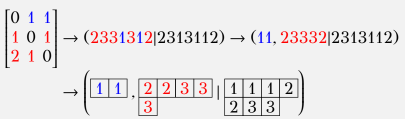

Figure 4: *Example of Filtered RSK for* $\mathbf{L}_{\{0,1,3\}|\{0,3\}}$

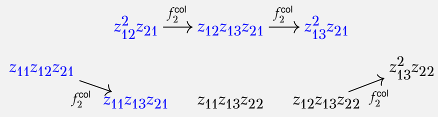

Figure 5: *Illustration of bicrystalline condition (Blue: standard monomials)*

Now, let $I \subseteq \mathbb{C}[\mathrm{Mat}_{m,n}]$ be an ideal fixed under the action of $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$. For any given term order $<$, the set of *standard monomials*

$$\mathrm{Std}_<(I) = \{\mathbf{m} \in \mathfrak{B} : \mathbf{m} \notin \mathrm{init}_<(I)\}$$

forms a basis for $\mathbb{C}[\mathrm{Mat}_{m,n}]/I$.

Definition. *An ideal I is bicrystalline for $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$ if there exists some term order $<$ under which the set $(\mathfrak{B}^{\mathbf{L}_{\mathbf{I}|\mathbf{J}}} \cap \mathrm{Std}_<(I)) \cup \{\emptyset\}$ is closed under the action of every admissible bicrystal operator for $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$.*

If I is bicrystalline, we obtain a combinatorial rule for computing $c^I_{\underline{\lambda}|\underline{\mu}}$.

Main Theorem (P.-Stelzer-Yong '24, arXiv:2403.09938). *If I is bicrystalline (as witnessed by a term order $<$), then*

$$c^I_{\underline{\lambda}|\underline{\mu}} = \#\left\{\mathbf{m} \in \mathfrak{B}^{\mathbf{L}_{\mathbf{I}|\mathbf{J}}} \cap \mathrm{Std}_<(\mathrm{I}) : \mathrm{filterRSK}(\mathbf{m}) = (T_{\underline{\lambda}}|T_{\underline{\mu}})\right\}$$

In principal, it requires infinitely many checks to tell whether an ideal I is bicrystalline; one must apply every bicrystal operator to every one of infinitely many standard monomials and check whether the resulting monomials are standard. However, in upcoming work, we show the following:

Theorem (P.-Stelzer-Yong '25+). *There exists a finite-time algorithm that determines whether any given ideal I with an action of some Levi group $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$ is bicrystalline for $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$.*

Main Example: Matrix Schubert Varieties

Our primary examples of bicrystalline ideals are ideals defining matrix Schubert varieties.

Proposition. *The matrix Schubert variety $\mathfrak{X}_w$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-stable with respect to the right action $(g,g')\cdot A = g^{-1}A(g'^{-1})^T$ whenever $\mathrm{Desc}_{\mathrm{row}}(w) \subseteq \mathbf{I}$ and $\mathrm{Desc}_{\mathrm{col}}(w) \subseteq \mathbf{J}$.*

Using the Knutson-Miller Grobner basis theorem ([2]), we show that the ideals defining matrix Schubert varieties are bicrystalline under antidiagonal term order.

Theorem (P.-Stelzer-Yong '24, arXiv:2403.09938). *Let $\mathfrak{X}_w \subseteq \mathrm{Mat}_{m,n}$ be a matrix Schubert variety and let $\mathrm{Desc}_{\mathrm{row}}(w) \subseteq \mathbf{I}$, $\mathrm{Desc}_{\mathrm{col}}(w) \subseteq \mathbf{J}$. Then the ideal $I(\mathfrak{X}_w)$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystalline.*

References

[1] Danilov, V. I.; Koshevoi, G. A.. Arrays and the combinatorics of Young tableaux. Russ. Math. Surv 60 (2005), 269-334.
[2] Knutson, Allen; Miller, Ezra. Gröbner geometry of Schubert polynomials. Ann. of Math. (2) 161 (2005), no. 3, 1245-1318.
[3] van Leeuwen, Marc. Double crystals of binary and integral matrices. Elec. J. of Combinatorics 13 (2006), no. 1, R86.

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