Counting homomorphisms in antiferromagnetic graphs via Lorentzian polynomials

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Abstract. We investigate inequalities involving graph homomorphisms into antiferromagnetic host graphs by establishing a novel connection with Lorentzian polynomials. This is **the first application** of the theory of Lorentzian polynomials to problems in extremal combinatorics.

Q. Fix a graph G and a graph class $\mathcal{H}$. Which $H \in \mathcal{H}$ maximises/minimises $\hom(H,G)^{1/e(H)}$?

Thm (Kahn '01 + Zhao '10). Among the d-regular graphs, $H = K_{d,d}$ maximises $\hom(H, \bullet\!-\!\bullet^{\circ})^{1/e(H)}$.

Thm (Zhao '10). $\forall H$, $\hom(H, \bullet\!-\!\bullet^{\circ})^2 \le \hom(H \times K_2, \bullet\!-\!\bullet^{\circ})$.

... $H = K_{d,d}$ maximises $\hom(H, K_q)^{1/e(H)}$.

Conj (Zhao '11). $\forall H, q$, $\hom(H, K_q)^2 \le \hom(H \times K_2, K_q)$.

Thm (Sah–Sawhney–Stoner–Zhao '20). holds even after attaching loops to K_q.

Conj (SSSZ '20). ... $H = K_{d,d}$ maximises $\hom(H,G)^{1/e(H)}$, whenever G is **antiferromagnetic**.

Conj (LOS). $\forall$ H and antiferromagnetic G, $\hom(H,G)^2 \le \hom(H \times K_2, G)$.

Def. A (edge-weighted) graph is **antiferromagnetic** if its adjacency matrix has at most one positive eigenvalue.

- includes $\bullet\!-\!\bullet^{\circ}$ and K_q possibly with loops.

A homogeneous polynomial is **Lorentzian** if

- it satisfies the 'partial derivative condition' and
- its support is M-convex.

Introduced indep. by (Brändén–Huh '20) and (Anari et al. '21).

- a powerful framework to describe negative correlation.
- essentially a 'higher-degree generalisation' of antiferromagnetic matrices.

Let f be a homogeneous polynomial of degree t in n variables. Its **complete homogeneous form** is $F_f\colon (\mathbb{R}^n)^t \to \mathbb{R}$ where

$$F_f(\mathbf{x}_1,\dots,\mathbf{x}_t) := \frac{1}{t!}\frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_t} f(\lambda_1\mathbf{x}_1+\cdots+\lambda_t\mathbf{x}_t).$$

Thm (Brändén–Huh '20). If f is Lorentzian, then $\forall \mathbf{x}_1 \in \mathbb{R}^n$ and $\mathbf{x}_2,\dots,\mathbf{x}_t \in \mathbb{R}^n_{\ge n}$,

$$F_f(\mathbf{x}_1,\mathbf{x}_1,\mathbf{x}_3,\dots,\mathbf{x}_t)\,F_f(\mathbf{x}_2,\mathbf{x}_2,\mathbf{x}_3,\dots,\mathbf{x}_t) \le F_f(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\dots,\mathbf{x}_t)^2.$$

Def. $h_H(-;G)\colon \mathbb{R}^n \to \mathbb{R}$: the G-**chromatic function of** H.

$$h_H(x_1,\dots,x_n;G) := \sum_{\phi\colon V(H)\to V(G)} \prod_{uv\in E(H)} G(\phi(u),\phi(v)) \prod_{v\in V(H)} x_{\phi(v)}$$

(A generalization of a chromatic symmetric polynomial)

$V_H(-;G)\colon (\mathbb{R}^n)^t \to \mathbb{R}$: the G-**volume of** H.

$$V_H(\mathbf{x}_1,\dots,\mathbf{x}_t;G) := \sum_{\phi\colon V(H)\to V(G)} \prod_{uv\in E(H)} G(\phi(u),\phi(v)) \prod_{u\in V(H)} x_{u,\phi(u)} = F_f(\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_t) \quad \text{for } f(-) = h_H(-;G)$$

$V_H(1_{A_1},\dots,1_{A_t};G) = \#$ of $H \to G$ s.t. $i \in V(H)$ maps into $A_i \subseteq V(G)$.

Thm (LOS). $h_{K_t}(\mathbf{x};G)$ is Lorentzian if G is antiferromagnetic.

Conj (LOS). $h_H(\mathbf{x};G)$ is Lorentzian $\forall$antiferromagnetic G $\iff H = K_t$.

Thm (LOS). The M-convex support condition for $h_H(\mathbf{x};G)$ to be Lorentzian $\forall$antiferromagnetic G: only need to check for $G = K_q$'s.

Cor (LOS). $\forall$antiferromagnetic G, $\forall \mathbf{a},\mathbf{b} \in (\mathbb{R}_{\ge 0})^n$,

$$V_{K_t}(\mathbf{a},\dots,\mathbf{a};G)V_{K_t}(\mathbf{b},\dots,\mathbf{b};G) \le V_{K_t}(\mathbf{a},\mathbf{b},\dots,\mathbf{b};G)V_{K_t}(\mathbf{b},\mathbf{a},\dots,\mathbf{a};G).$$

Thm (LOS). (a) $\forall$antiferromagnetic G, $\forall A,B \subseteq V(G)$,

$$\hom(K_t, G[A])\hom(K_t,G[B]) \le \hom_{\mathrm{b}}(K_t \times K_2, G[A,B]).$$

(b) $\forall H \in \{\text{paths, even cycles, complete multipartite}\}$, $\forall A,B \subseteq V(K_q)$,

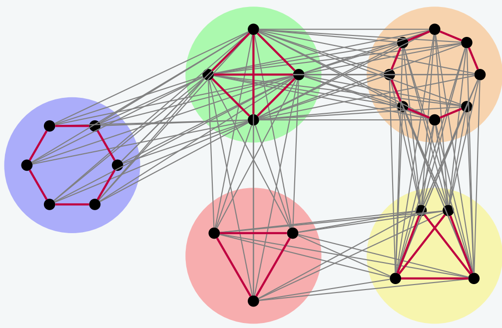
$$\hom(H, K_q[A])\hom(H,K_q[B]) \le \hom_{\mathrm{b}}(H \times K_2, K_q[A,B]).$$

Q. Can the hands-on proof of (b) be reformulated using the theory of Lorentzian polynomials and extended to include more H?

Thm (LOS). Construct $\mathcal{H}_1, \mathcal{H}_2$ consisting of various new graphs s.t.

(a) $H \in \mathcal{H}_1 \implies \hom(H,G)^2 \le \hom(H\times K_2, G)$ $\forall$antiferromagnetic G;

(b) $H \in \mathcal{H}_2 \implies \hom(H,K_q)^2 \le \hom(H\times K_2, K_q)$ $\forall q$.



$\in \mathcal{H}_2$

Check the preprint at arXiv:2506.13659

2025.7 FPSAC Hokkaido University, Sapporo