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On a conjecture of Dyer on the join in the weak order of a Coxeter group

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Coxeter groups background

- **Coxeter system** (W,S) : $W =< S \mid R >=< s_1,\dots,s_n \mid (s_is_j)^{m_{ij}} = e >$, where $m_{ii}=1$ and $m_{ij}=m_{ji}\geq 2$, if $i\neq j$;
- **Reflections**: $T=\{wsw^{-1}\mid w\in W, s\in S\}$, $S=\{\textit{simple reflections}\}$;
- **Length** of $w\in W$: $\ell(w)=\min\{k\in\mathbb{N}\mid w=s_{i_1}s_{i_2}\cdots s_{i_k},\ s_{i_j}\in S\}$;
- **Left-reflection set** of $w\in W$: $T_L(w)=\{t\in T\mid \ell(tw)<\ell(w)\}$;
- **Bruhat graph** $B(W)$: the directed graph having W as vertex set and for any $u,v\in W$, an edge $u\xrightarrow{t}v$ if and only if there is $t\in T$ such that $v=tu$ and $\ell(u)<\ell(v)$;
- **Weak order** $(W,\leq_R)$: $u\leq_R v$ if there are $s_{i_1},s_{i_2},\dots,s_{i_k}\in S$ such that
  $-v=us_{i_1}\cdots s_{i_k}$;
  $-\ell(us_{i_1}\cdots s_{i_j})<\ell(us_{i_1}\cdots s_{i_j}s_{i_{j+1}})$, for any $j\in\{1,2,\dots,k-1\}$.
  Remark: left-reflection sets characterize the weak order:
  $$u\leq_R v \iff T_L(u)\subseteq T_L(v).$$

Type A Coxeter groups

- The Coxeter group of type A_{n-1} is isomorphic to the Symmetric group S_n with generators $\{s_1,\dots,s_{n-1}\}$, where $s_i=(i\ i+1)$ and their relations are
  $$s_i^2=e=(s_is_{i+1})^3.$$
- Reflections of S_n coincide with transpositions:
  $$T=\{(a\ b)\mid 1\leq a<b\leq n\}.$$
- The left-reflection set of a permutation $\sigma\in S_n$ is given by
  $$T_L(\sigma)=\{(a\ b)\in T\mid a<b,\ \sigma^{-1}(a)>\sigma^{-1}(b)\}.$$

Example

As a Coxeter group S_4 is generated by $S=\{(1\ 2),(2\ 3),(3\ 4)\}$ and its reflections are $T=\{(1\ 2),(2\ 3),(3\ 4),(1\ 3),(2\ 4),(1\ 4)\}$. In Figures 1 and 2 we compare $(S_4,\leq_R)$ and $B(S_4)$.

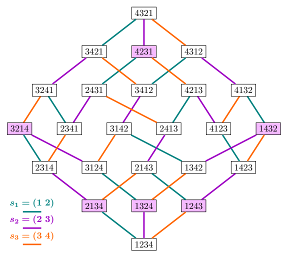

Fig. 1: Hasse diagram of $(S_4,\leq_R)$

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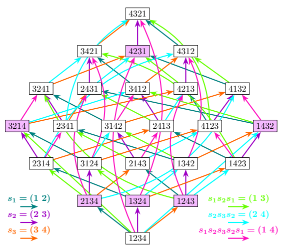

Fig. 2: Bruhat graph $B(S_4)$

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(u,v)-Bruhat path

Given two elements $u,v\in W$; a (u,v)-*Bruhat path* is any (directed) path in $B(W)$ starting from the vertex e and whose edges have labels in the set $T_L(u)\cup T_L(v)$.
We denote by $V_W(u,v)$ the set of vertices of all the (u,v)-Bruhat paths in $B(W)$.

The conjecture

Conjecture (Dyer, [2])

Let W be a finite Coxeter group and $u,v\in W$. Then
$$T_L(u\vee_R v)=T\cap V_W(u,v).$$

Remark: this conjecture states that the left-reflection set of the join $u\vee_R v$ in the poset $(W,\leq_R)$ is the set of reflections reached by all possible (u,v)-Bruhat paths.

Example

- Consider $\sigma=3124,\tau=1423\in S_4$;
- $T_L(\sigma)=\{(1\ 3),(2\ 3)\}$, $T_L(\tau)=\{(2\ 4),(3\ 4)\}$;
- from the Hasse diagram in Fig. 1, observe that
  $$\sigma\vee_R\tau=4312;$$
- using the labels in $T_L(\sigma)\cup T_L(\tau)$ compute all the (σ,τ)-Bruhat paths in Fig. 3, where reflections are highlighted;
- compute
  $$T\cap V_{S_4}(\sigma,\tau)=\{(1\ 3),(1\ 4),(2\ 3),(2\ 4),(3\ 4)\}$$
  and check that is equal to $T_L(\sigma\vee_R\tau)$.

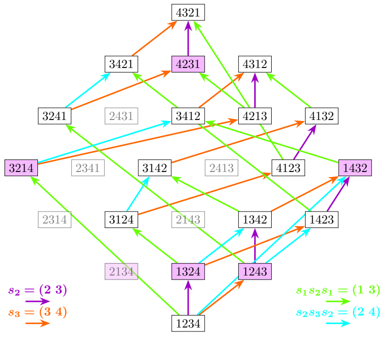

Fig. 3: All (σ,τ)-Bruhat paths.

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How did this conjecture arise?

- In [2], Dyer defined the *extended weak order* of a Coxeter group, a bounded poset that generalizes the weak order $(W,\leq_R)$, and conjectures that:
  1. the extended weak order is a lattice for any Coxeter group;
  2. there is a characterization of the join in the extended weak order.
- Conjecture 1 was proven in the affine case by Barkley and Speyer in [1].
- Conjecture 2 is open even in the case of finite Coxeter groups. In these cases, it can be stated with the above formulation that was told to us by Hohlweg [3].

Main result

Theorem

The conjecture holds for Coxeter groups of type I_2, A, F_4, H_3.

We checked the cases F_4 and H_3 with the open-source software SageMath [5].

Idea of proof for type A

Theorem (e.g. [4])

Let $\sigma,\tau\in S_n$ and J^{tc} denote the transitive closure of $J\subseteq T$; then
$$T_L(\sigma\vee_R\tau)=(T_L(\sigma)\cup T_L(\tau))^{tc}.$$

Remark: J is transitively closed if for $(i\ j),(j\ k)\in J \implies (i\ k)\in J$.

- $(T_L(\sigma)\cup T_L(\tau))^{tc}\subseteq T\cap V_{S_n}(\sigma,\tau)$ is proven by showing that for any $(a\ b)\in(T_L(\sigma)\cup T_L(\tau))^{tc}$, there is a *palindromic* (σ,τ)-Bruhat path
  $$e\xrightarrow{(a\ i_1)}(a\ i_1)\xrightarrow{(i_1\ i_2)}\cdots\cdots\xrightarrow{(i_1\ i_2)}(a\ i_1)(a\ b)\xrightarrow{(a\ i_1)}(a\ b).$$
- To prove $(T_L(\sigma)\cup T_L(\tau))^{tc}\supseteq T\cap V_{S_n}(\sigma,\tau)$ we use the following

Lemma

All the edges of a Bruhat path from e to $(a\ b)\in T$ are labeled by reflections in $\{(i\ j)\mid a\leq i<j\leq b\}$.

- We argue recursively on the edges of a (σ,τ)-Bruhat path that reaches $(a\ b)$ to show that there is a chain $a=i_0<i_1<\cdots<i_{k-1}<i_k=b$, such that $(i_{r-1}\ i_r)\in T_L(\sigma)\cup T_L(\tau)$, for any $r\in[k]$; i.e. $(a\ b)\in(T_L(\sigma)\cup T_L(\tau))^{tc}$.

Concluding remarks

- For other Coxeter groups: in type B, we have made some progress by adapting the combinatorial approach that was successful in type A.
- Interestingly, the statement of the conjecture is not trivial to prove even in the particular case $u\leq_R v$, in which it can be reformulated as

Special case

Let $w\in W$, then a reflection given by a length-increasing product of elements of $T_L(w)$ is itself an element of $T_L(w)$.

References

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