Deforming the motivic Segre classes of Schubert cells in the Grassmannian

Raj Gandhi, Cornell University

Cotangent bundles of Grassmannians

The **Grassmannian** $\mathrm{Gr}(d,n)$ is the variety of d-dimensional subspaces in $\mathbb{C}^n$. The cotangent bundle $T^*(\mathrm{Gr}(d,n))$ of $\mathrm{Gr}(d,n)$ has a rich geometric structure: it has an action of a $(n+1)$-torus $\widehat{T} := (\mathbb{C}^\times)^{n+1}$, and it is an example of a symplectic resolution. Maulik and Okounkov ([4], [5]) defined natural bases in localizations $H^{\mathbf{loc}}_{\widehat{T}}(T^*(\mathrm{Gr}(d,n)))$ and $K^{\mathbf{loc}}_{\widehat{T}}(T^*(\mathrm{Gr}(d,n)))$ of the $\widehat{T}$-equivariant cohomology ring and $\widehat{T}$-equivariant K-ring of $T^*(\mathrm{Gr}(d,n))$. In $H^{\mathbf{loc}}_{\widehat{T}}(T^*(\mathrm{Gr}(d,n)))$, certain quotients of the basis elements are called **Segre-Schwartz-MacPherson (SSM) classes** of Schubert cells, while in $K^{\mathbf{loc}}_{\widehat{T}}(T^*(\mathrm{Gr}(d,n)))$, certain quotients of the basis elements are called **motivic Segre classes** of Schubert cells. Let Γ be the set of 01 sequences $\lambda = \lambda_1\lambda_2\cdots\lambda_n$, where 1 appears d times and 0 appears $n-d$ times in the sequence. The basis in either ring is indexed by Γ. We will denote the basis elements in either ring by S_λ.

Getting a hold of the S_λ combinatorially

The fixed points of the $\widehat{T}$-action on $T^*(\mathrm{Gr}(d,n))$ are indexed by Γ. Let h be either singular cohomology H or K-theory K. The **localization map**

$$\iota\colon h_{\widehat{T}}(T^*(\mathrm{Gr}(d,n))) \to \bigoplus_{\lambda\in\Gamma} h_{\widehat{T}}(\mathrm{pt}),$$

defined by restricting $h_{\widehat{T}}(T^*(\mathrm{Gr}(d,n)))$ to the corresponding fixed point $\lambda\in\Gamma$, is injective.

We will now describe the image of S_λ under ι. Recall that W is generated by simple transpositions $r_1,\dots,r_{n-1}$, where r_i swaps i and $i+1$ and fixes all other j. We define the element $\omega := 1^d0^{n-d}$ of Γ.

1. $H^{\mathbf{loc}}_{\widehat{T}}(T^*(\mathrm{Gr}(d,n)))$. In this case $H^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$ is the localization of $\mathbb{Z}[y_1,\dots,y_n,\hbar]$ at $\{y_i-y_j-\hbar\}_{1\le i\ne j\le n}$. Below are the formulas for S_ω restricted to fixed points $\lambda\in\Gamma$:
$$S_\omega|_\lambda = \begin{cases}\prod_{i>j:\lambda_i<\lambda_j}\frac{y_i-y_j}{y_i-y_j-\hbar}, & \lambda=\omega;\\ 0, & \lambda\ne\omega.\end{cases}$$
There is a natural action of W on the ring $\mathbb{Z}[y_1,\dots,y_n,\hbar]$, defined on generators by $w(y_i) := y_{w(i)}$ and $w(\hbar) := \hbar$ for all $w\in W$, and this action extends to $H^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$. This W-action also extends to the ring $\bigoplus_{\lambda\in\Gamma} H^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$ by $w\cdot(f_\lambda)_\lambda := (w(f_\lambda))_{w(\lambda)}$ for all $w\in W$. Consider the $\mathbb{Z}[\hbar]$-linear operators on $H^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$:
$$\partial_i(f) = \frac{\hbar}{y_i-y_{i+1}}f + \frac{y_i-y_{i+1}-\hbar}{y_i-y_{i+1}}r_i(f),\quad f\in H^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt}).$$
The operators ∂_i naturally define $\mathbb{Z}[\hbar]$-linear operators on $\bigoplus_{\lambda\in\Gamma} H^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$. We have
$$\iota(S_{w^{-1}(\omega)}) = \partial_w(\iota(S_\omega)).$$
2. $K^{\mathbf{loc}}_{\widehat{T}}(T^*(\mathrm{Gr}(d,n)))$. In this case $K^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$ is the localization of $\mathbb{Z}[e^{\pm y_1},\dots,e^{\pm y_n},q^{\pm1}]$ at $\{1-q^2e^{y_i-y_j}\}_{1\le i\ne j\le n}$. Below are the formulas for S_ω restricted to fixed points $\lambda\in\Gamma$:
$$S_\omega|_\lambda = \begin{cases}\prod_{i>j:\lambda_i<\lambda_j}\frac{1-e^{y_i-y_j}}{1-q^2e^{y_i-y_j}}, & \lambda=\omega;\\ 0, & \lambda\ne\omega.\end{cases}$$
There is a natural action of W on $\mathbb{Z}[e^{\pm y_1},\dots,e^{\pm y_n},q^{\pm1}]$, defined on generators by $w(e^{\pm y_1}) := e^{\pm y_{w(i)}}$ and $w(q^2) := q^2$ for all $w\in W$, and this action extends to $K^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$. This W-action also extends to the ring $\bigoplus_{\lambda\in\Gamma} K^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$ by $w\cdot(f_\lambda)_\lambda := (w(f_\lambda))_{w(\lambda)}$ for all $w\in W$. Consider the $\mathbb{Z}[q^{\pm1}]$-linear operators on $K^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$:
$$\partial_i(f) = \frac{1-q^2}{1-e^{y_{i+1}-y_i}}f + \frac{1-q^2e^{y_i-y_{i+1}}}{1-e^{y_i-y_{i-1}}}r_i(f),\quad f\in K^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt}).$$
The operators ∂_i naturally define $\mathbb{Z}[q^{\pm1}]$-linear operators on $\bigoplus_{\lambda\in\Gamma} K^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$. We have
$$\iota(S_{w^{-1}(\omega)}) = \partial_w(\iota(S_\omega))$$

Deforming the motivic Segre classes

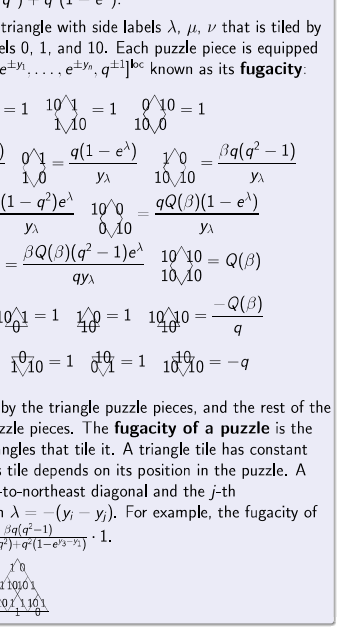
Consider the following $\mathbb{Z}[\beta][q^{\pm1}]$-linear operators on $\mathbb{Z}[\beta][e^{\pm y_1},\dots,e^{\pm y_n},q^{\pm1}]$:
$$\partial_i := \frac{\beta(1-q^2)}{1-e^{y_{i+1}-y_i}} + \frac{\beta(1-q^2)+q^2(1-e^{y_i-y_{i-1}})}{1-e^{y_i-y_{i-1}}}r_i.$$
The $\beta=1$ specialization is the operator on $K^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$. The $\beta\to0$ limit (for some definition of 'limit') is the operator on $H^{\mathbf{loc}}_{\widehat{T}}(\mathrm{pt})$. The operators ∂_i satisfy the following relations:

1. $\partial_i\circ\partial_j = \partial_j\circ\partial_i$ for all $i,j=1,\dots,n-1$ such that $|i-j|>1$.
2. $\partial_j\circ\partial_{i+1}\circ\partial_i = \partial_{i+1}\circ\partial_i\circ\partial_{i+1}$ for all $i=1,\dots,n-2$.
3. $(\partial_i+q^2)\circ(\partial_i+(q^2\beta-q^2-\beta)) = 0$ for all $i=1,\dots,n-1$.

The operators ∂_i also act on the localization
$$\mathbb{Z}[\beta][e^{\pm y_1},\dots,e^{\pm y_n},q^{\pm1}]^{\mathrm{loc}}$$
of $\mathbb{Z}[\beta][e^{\pm y_1},\dots,e^{\pm y_n},q^{\pm1}]$ at the elements $\{\beta(1-q^2)+q^2(1-e^{y_i-y_j})\}_{1\le i\ne j\le n}$. We will now define elements S_λ in
$$R^{\mathrm{loc}} := \bigoplus_{\lambda\in\Gamma}\mathbb{Z}[\beta][e^{\pm y_1},\dots,e^{\pm y_n},q^{\pm1}]^{\mathrm{loc}}.$$
The Weyl group W acts on R^{loc} by $w\cdot(f_\lambda)_\lambda := (w(f_\lambda))_{w(\lambda)}$, $w\in W$. The element S_ω is
$$S_\omega|_\lambda = \begin{cases}\prod_{i>j:\lambda_i<\lambda_j}\frac{1-e^{y_i-y_j}}{\beta(1-q^2)+q^2(1-e^{y_i-y_j})}, & \text{if } \lambda=\omega;\\ 0, & \text{otherwise.}\end{cases}$$
The other S_λ are defined by the rule $S_{w^{-1}(\omega)} := \partial_w(S_\omega)$. At $\beta=1$, the S_λ are the motivic Segre classes of Schubert cells. At $\beta\to0$, the S_λ are the SSM classes of Schubert cells.

Knutson-Tao puzzles

Define $Q(\beta) := q^2+\beta-q^2\beta$. For $\lambda\in\mathbb{Z}[y_1,\dots,y_n]$, we will set
$$y_\lambda := \beta(1-q^2)+q^2(1-e^\lambda).$$
A **puzzle** with side labels λ, μ, ν in Γ is a triangle with side labels λ, μ, ν that is tiled by the following **puzzle pieces** with edge labels 0, 1, and 10. Each puzzle piece is equipped with a function from $\mathbb{Z}[y_1,\dots,y_n]$ to $\mathbb{Z}[\beta][e^{\pm y_1},\dots,e^{\pm y_n},q^{\pm1}]^{\mathrm{loc}}$ known as its **fugacity**:



$$\frac{\beta(1-q^2)}{y_\lambda}\quad \frac{\beta(1-q^2)}{y_\lambda}\quad \frac{q(1-e^\lambda)}{y_\lambda}\quad \frac{\beta q(q^2-1)}{y_\lambda}$$
$$\frac{\beta(1-q^2)e^\lambda}{y_\lambda}\quad \frac{\beta(1-q^2)e^\lambda}{y_\lambda}\quad \frac{qQ(\beta)(1-e^\lambda)}{y_\lambda}$$
$$\frac{qQ(\beta)(1-e^\lambda)}{y_\lambda}\quad \frac{\beta Q(\beta)(q^2-1)e^\lambda}{qy_\lambda}\quad Q(\beta)$$
$$\frac{-Q(\beta)}{q}$$

The bottom row of a puzzle is always tiled by the triangle puzzle pieces, and the rest of the puzzle is tiled by (not rotated) rhombus puzzle pieces. The **fugacity of a puzzle** is the product of fugacities of the rhombi and triangles that tile it. A triangle tile has constant fugacity, whereas the fugacity of a rhombus tile depends on its position in the puzzle. A rhombus tile that lies in the i-th southwest-to-northeast diagonal and the j-th northwest-to-southeast diagonal depends on $\lambda = -(y_i-y_j)$. For example, the fugacity of the puzzle below is $\frac{qQ(\beta)(1-e^{y_2-y_1})}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}\cdot\frac{\beta q(q^2-1)}{\beta(1-q^2)+q^2(1-e^{y_3-y_1})}\cdot 1$.

Multiplying the deformed classes

We will use the notation $\lambda\nu\mu$ to mean the sum over the fugacities of all possible puzzles with the prescribed boundary labels. The **length** $l(\lambda)$ of an element $\lambda\in\Gamma$ is the minimal number of simple transpositions needed to turn λ into $0^{n-d}1^d$. The **homogenization** of S_λ by a factor q is $q^{l(\lambda)}S_\lambda$. We can now state our main theorem:

THEOREM.
The product of two classes $q^{l(\lambda)}S_\lambda$ and $q^{l(\mu)}S_\mu$ is given by the "puzzle" formula
$$(q^{l(\lambda)}S_\lambda)(q^{l(\mu)}S_\mu) = \sum_\nu \lambda\nu\mu\,(q^{l(\nu)}S_\nu).$$

As in [3], a **positivity monoid** M is a submonoid of $\mathbb{Z}[\beta][e^{\pm y_1},\dots,e^{\pm y_n},q^{\pm1}]^{\mathrm{loc}}$ under addition such that $M\cap(-M)=0$. Consider the submonoid M of $\mathbb{Z}[\beta][e^{\pm y_1},\dots,e^{\pm y_n},q^{\pm1}]^{\mathrm{loc}}$, defined as the set of sums of products of the factors over all $1\le i<j\le n$:
$$-q^{\pm}\qquad Q(\beta)\qquad e^{y_j-y_i}\qquad \frac{\beta(1-q^2)}{\beta(1-q^2)+q^2(1-e^{y_j-y_i})}\qquad -\frac{1-e^{y_j-y_i}}{\beta(1-q^2)+q^2(1-e^{y_j-y_i})}$$
Then M is a positivity monoid of $\mathbb{Z}[\beta][e^{\pm y_1},\dots,e^{\pm y_n},q^{\pm1}]^{\mathrm{loc}}$. As the structure constants for $\{q^{l(\lambda)}S_\lambda\}_{\lambda\in\Gamma}$ lie in the positivity monoid M, it is in this sense that we consider the structure constants and puzzle formula for them **positive**.

Example

Let us compute the classes $q\cdot S_{10}$ and S_{01} for $T^*(\mathrm{Gr}(1,2))$. The classes are
$$q\cdot S_{10} = q\cdot\left[0,\frac{1-e^{y_2-y_1}}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}\right];\qquad S_{01} = (\tfrac{1}{q}\partial_\sigma)(q\cdot S_{10}) = \left[1,\frac{\beta(1-q^2)}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}\right].$$
Let us now multiply classes together:
$$(q\cdot S_{10})^2 = \frac{q(1-e^{y_2-y_1})}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}(q\cdot S_{10});\quad (q\cdot S_{10})S_{01} = S_{01}(q\cdot S_{10}) = \frac{\beta(1-q^2)}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}(q\cdot S_{10});$$
$$S_{01}^2 = S_{01} + \frac{\beta q(q^2-1)}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}(q\cdot S_{10}).$$
The puzzles give the same result:

$$\frac{q(1-e^{y_2-y_1})}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})};\qquad \frac{\beta(1-q^2)}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})};$$
$$\frac{\beta(1-q^2)}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})};\qquad 1;$$
$$\frac{\beta q(q^2-1)}{\beta(1-q^2)+q^2(1-e^{y_2-y_1})}.$$

References

[1] Allen Knutson and Terence Tao. Puzzles and (equivariant) cohomology of Grassmannians. *Duke Math. J.*, 119(2):221-260, 2003.

[2] Allen Knutson and Paul Zinn-Justin. Schubert puzzles and integrability I: Invariant trilinear forms. 2024.

[3] Allen Knutson and Paul Zinn-Justin. Schubert puzzles and integrability II: Multiplying motivic Segre classes. 2024.

[4] Davesh Maulik and Andrei Okounkov. Quantum groups and quantum cohomology. *Astérisque*, (408):ix+209, 2019.

[5] Andrei Okounkov. Lectures on K-theoretic computations in enumerative geometry. 2017.

[6] Paul Zinn-Justin. Littlewood-Richardson coefficients and integrable tilings. *Electron. J. Combin.*, 16(1):Research Paper 12, 33, 2009.