Faces of Parking Function Polytopes

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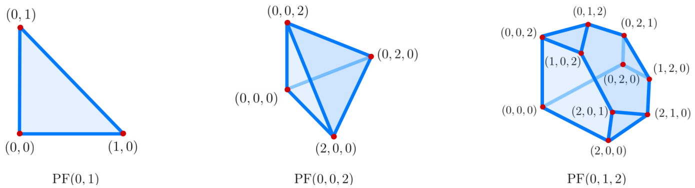
Parking Function Polytopes Suppose that $\mathbf{u} \in \mathbb{R}^n$ is a vector satisfying $0 \le u_1 \le \cdots \le u_n$. Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n_{\ge 0}$ and $b_1 \le b_2 \le \cdots \le b_n$ be the non-decreasing rearrangement of $a_1, \dots, a_n$. We say that $\mathbf{a}$ is a $\mathbf{u}$-*parking function* if $b_i \le u_i$ for all $i = 1, \dots, n$. The *parking function polytope* $\mathrm{PF}(\mathbf{u})$ is defined to be the convex hull of all $\mathbf{u}$-parking functions.



Definition 1. Suppose that there are ℓ positive integers appearing in $\mathbf{u}$: $d_1 < d_2 < \cdots < d_\ell$. We define $m_0(\mathbf{u})$ to be the number of 0's in $\mathbf{u}$, and $m_i(\mathbf{u})$ be the number of d_i's in $\mathbf{u}$ for each $1 \le i \le \ell$. The *multiplicity vector* and the *data vector* of $\mathbf{u}$ are defined respectively to be $\mathbf{m}(\mathbf{u}) = (m_0(\mathbf{u}), m_1(\mathbf{u}), \dots, m_\ell(\mathbf{u}))$ and $\mathbf{d}(\mathbf{u}) = (d_1, d_2, \dots, d_\ell)$. We refer to $(\mathbf{m}, \mathbf{d})$ as the *MD pair* for $\mathbf{u}$.

For example, if $\mathbf{u} = (0, 0, 4, 4, 4, 6, 8, 8)$, then $\mathbf{m}(\mathbf{u}) = (2, 3, 1, 2)$ and $\mathbf{d}(\mathbf{u}) = (4, 6, 8)$. With the notion of MD pair, we will henceforth write $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ interchangeably with $\mathrm{PF}(\mathbf{u})$ throughout the remainder of the poster.

$\mathrm{PF}(\mathbf{m}, \mathbf{d})$ and Others It can be shown that parking function polytopes can be "realized" as the following polytopes:

- (Type A) Generalized permutohedra
- Type B generalized permutohedra
- Polymatroids

Thus, theories about these families of polytopes can be applied to parking function polytopes.

Skewed Binary Partitions We introduce here an analogue of ordered partition called "skewed binary partition" of the set $[0, n]$. Its blocks are separated into two different kinds–*homogeneous* and *non-homogeneous*–and maybe empty. This combinatorial object allows us to describe the normal fan (and hence the face poset) of parking function polytope.

Definition 2. Let $k \in \mathbb{Z}_{\ge 0}$. A *skewed binary partition* of $[0, n]$ into $k + 2$ blocks is an ordered tuple $(B_{-1}, B_0, \dots, B_k)$ of disjoint subsets of $[0, n]$ such that $B_{-1} \sqcup B_0 \sqcup B_1 \sqcup \cdots \sqcup B_k = [0, n]$ satisfying:

(1) B_0 is homogeneous, provided $|B_0| \ge 2$, and B_{-1} is non-homogeneous.
(2) $0 \in B_{-1}$ or $0 \in B_0$. If $0 \in B_{-1}$, then B_{-1} contains at least another element and $B_0 = \emptyset$. Hence, if $0 \in B_{-1}$, then $|B_{-1}| \ge 2$ and $|B_0| = 0$.
(3) For each $0 \le i \le k$, if B_i is a singleton, then it is non-homogeneous.
(4) $B_i \ne \emptyset$ for all $1 \le i \le k$.

skewed binary partition $\mathcal{B}$ of $[0, 8]$
$(\{0, 2, 5\}, \emptyset, \{6, 7\}, \{1, 3, 4, 8\}^\star)$
$(\{2, 5\}, \{0, 7\}^\star, \{6\}, \{1, 3, 4, 8\}^\star)$
$(\{1, 3, 4, 5, 8\}, \{0\}, \{2\}, \{6, 7\})$
$(\emptyset, \{0\}, \{2, 3, 8\}, \{1, 6, 7\}^\star, \{4, 5\})$
$(\emptyset, \{0, 1, 2, 3, 4, 5, 6, 7, 8\}^\star)$

Definition 3. For each skewed binary partition $\mathcal{B} = (B_{-1}, B_0, B_1, \dots, B_k)$ of $[0, n]$, we associate the preorder $\preceq_{\mathcal{B}}$ on the set $[0, n]$ by letting

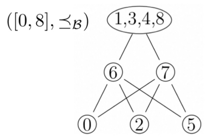
$$p \preceq_{\mathcal{B}} q \quad \text{if } p \in B_i \text{ and } q \in B_j \text{ and } i < j$$
$$p \equiv_{\mathcal{B}} q \quad \text{if } p, q \in B_i \text{ for some homogeneous block } B_i.$$

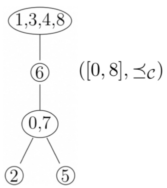
A binary partition $\mathcal{C}$ is said to be a *contraction* of another binary partition $\mathcal{B}$, denoted by $\mathcal{C} \le \mathcal{B}$, if $\preceq_{\mathcal{C}}$ is a contraction of $\preceq_{\mathcal{B}}$.

Definition 4. Let $\preceq$ be a preorder on $[0, n]$. The *sliced preorder cone* $\tilde{\sigma}_{\preceq}$ associated to $\preceq$ is given by
$\tilde{\sigma}_{\preceq} := \{(c_1, \dots, c_n) \mid c_0 = 0 \text{ and } c_i \le c_j \text{ if } i \preceq j, \text{ for } i, j \in [0, n]\}.$

Let $\mathcal{B}$ and $\mathcal{C}$ be the first two skewed binary partitions in the table above. Then the preposets $([0, 8], \preceq_{\mathcal{B}})$ and $([0, 8], \preceq_{\mathcal{C}})$, and $\tilde{\sigma}_{\preceq_{\mathcal{B}}}, \tilde{\sigma}_{\preceq_{\mathcal{C}}}$ are shown below.

$\{0, c_2, c_5 \le c_6, c_7 \le c_1 = c_3 = c_4 = c_8\}$





$\{c_2, c_5 \le 0 = c_7 \le c_6 \le c_1 = c_3 = c_4 = c_8\}$

Theorem 1. *Let $(\mathbf{m}, \mathbf{d})$ be an MD pair. Then there exists a poset of skewed binary partitions $\mathcal{SBP}(\mathbf{m})$ ordered by contraction that is isomorphic to the normal fan of $\mathrm{PF}(\mathbf{m}, \mathbf{d})$. Moreover, if $\mathbf{F}_{\mathcal{B}}$ is the face of $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ corresponding to the skewed binary partition $\mathcal{B}$, then $\mathrm{ncone}(\mathbf{F}_{\mathcal{B}}, \mathrm{PF}(\mathbf{m}, \mathbf{d})) = \tilde{\sigma}_{\mathcal{B}}$. Hence, the face poset of $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is isomorphic to the dual of $\mathcal{SBP}(\mathbf{m})$.*

Corollary 1. *The combinatorial types of parking function polytopes depend solely on the multiplicity vector, i.e. two parking functions polytopes $\mathrm{PF}(\mathbf{u}_1)$ and $\mathrm{PF}(\mathbf{u}_2)$ have isomorphic face posets if $\mathbf{m}(\mathbf{u}_1) = \mathbf{m}(\mathbf{u}_2)$.*

There are other results in our paper that cannot be put on this poster due to limited space. Here are a few highlighted results:

- A characterization of the contractions of skewed binary partitions in terms of bipartite graphs.
- The dimentions of the normal cone corresponding to $\mathcal{B}$.
- A characterization of the "types" of skewed binary partitions in $\mathcal{SBP}(\mathbf{m})$.
- Ehrhart polynomials and volume.

h-Polynomials Not all $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ are simple. The following result provides a characterization of those that are.

Corollary 2. *Let $(\mathbf{m}, \mathbf{d})$ be an MD pair where $\mathbf{m} = (m_0, m_1, \dots, m_\ell)$. Then $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is simple if and only if either $\mathbf{m} = (0, n)$ or $(n-1, 1)$ or $m_1 = \cdots = m_{\ell-1} = 1$ for some $\ell \ge 2$.*

Definition 5. For $p, q \in \mathbb{N}$, let $T(p, q)$ be the poset on $[p + q]$ defined by the covering relations $j \lessdot j + 1$ for all $j \in [p - 1]$ and $p \lessdot k$ for all $k \in [p+1, q]$. Let $\mathfrak{S}_n(T(p, q)) := \{\sigma(T(p, q)) \mid \sigma \in \mathfrak{S}_n\}$ be the set of all posets on $[p + q]$ having the same Hasse diagram.

Definition 6. Given a poset $(T(p, q), \le_T)$ on $[p + q]$, we define its *generalized Eulerian polynomial* to be

$$A(T(p, q), t) := \sum_{T \in \mathfrak{S}_n(T(p,q))} t^{\mathrm{des}(T)}$$

where $\mathrm{des}(T)$ is the number of descents in the poset T, i.e., the number of (i, j) such that $i \lessdot_T j$ and $j < i$ (usual order on $\mathbb{R}$).

Theorem 2. *Let $(\mathbf{m}, \mathbf{d})$ be an MD pair where $\mathbf{m} = (m_0, m_1, \dots, m_\ell)$ for some $\ell \ge 2$. Suppose that $\mathrm{PF}(\mathbf{m}, \mathbf{d})$ is n-dimensional and simple. Then its h-polynomial is*

$$h(t) = \begin{cases} s_1(t) + t \sum_{i=1}^{\ell-1} \binom{n}{i+m_\ell} A(T(i, m_\ell), t) & \text{if } m_0 = 0 \\ s_2(t) + t \sum_{i=1}^{\ell-2} \binom{n}{i+m_\ell} A(T(i, m_\ell), t) & \text{otherwise} \end{cases}$$

where $s_1(t)$ and $s_2(t)$ are polynomials with non-negative integral coefficients of degree at most n.

Note that we can also express $A(T(p, q), t)$ in terms of Eulerian polynomials. Thus, $h(t)$ can be expressed in terms of Eulerian polynomials as well. For instance, the h-polynomials of $\mathrm{PF}(1, \dots, n)$ and $\mathrm{PF}(0, \dots, n-1)$ respectively equal

$$1 + \sum_{k=1}^{n} \binom{n}{k} t A_k(t) \text{ and } 1 + tA_n(t) + \sum_{k=1}^{n-2} \binom{n}{k} t A_k(t).$$

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