Tutte polynomials in superspace

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Abstract

We associate a **quotient of superspace** to any **hyperplane arrangement** by considering the **differential closure** of a **power ideal** (a particular ideal generated by powers of homogeneous linear forms). This quotient is a superspace analogue of the external **zonotopal algebra** and also contains the central zonotopal algebra. We show that the bigraded **Hilbert series** of this quotient is equal to a transformation of the **Tutte polynomial**. We also construct an explicit basis for the Macaulay inverse. These results generalize work of Ardila–Postnikov and Holtz–Ron.

Objects of study

Power ideals

A *(linear) hyperplane* H is a codimension one subspace of $\mathbb{C}^n$. An *arrangement* $\mathcal{A}$ of hyperplanes is a collection $\{H_1, \ldots, H_m\}$ of hyperplanes where **repeats are allowed**. For any line $L = \mathbb{C} \cdot (\ell_1, \ldots, \ell_n)$ in $\mathbb{C}^n$, let

$$\lambda_L := \ell_1 x_1 + \cdots + \ell_n x_n.$$

For an integer $k \geq -1$, Ardila–Postnikov define the *power ideal*

$$J_{\mathcal{A},k} := \left(\lambda_L^{\rho_{\mathcal{A}}(L)+k} \mid L \subseteq \mathbb{C}^n \text{ a line} \right)$$

where

$$\rho_{\mathcal{A}}(L) := \#\{1 \leq i \leq m \mid L \not\subseteq H_i\}.$$

The cases $k \in \{-1, 0, 1\}$ are of particular interest, as the respective quotients $\mathbb{C}[\boldsymbol{x}_n]/J_{\mathcal{A},k}$ are the *internal, central* and *external zonotopal algebras*. The (singly-graded) Hilbert series of these algebras are obtained as univariate specializations of the corresponding Tutte polynomial. More precisely, if r is the rank of $\mathcal{A}$,

$$\mathrm{Hilb}(\mathbb{C}[\boldsymbol{x}_n]/J_{\mathcal{A},0}; q) = q^{m-r} T_{\mathcal{A}}(1, q^{-1}),$$
$$\mathrm{Hilb}(\mathbb{C}[\boldsymbol{x}_n]/J_{\mathcal{A},1}; q) = q^{m-r} T_{\mathcal{A}}\left(1+q, q^{-1}\right).$$

Superspace

Superspace of rank n is the tensor product $\Omega_n := \mathbb{C}[\boldsymbol{x}_n] \otimes \wedge\{\boldsymbol{\theta}_n\}$, which can also be thought of as the space of regular (polynomial-valued) differential forms on $\mathbb{C}^n$. This space comes with the *exterior derivative* $d : \Omega_n \to \Omega_n$ defined by

$$d\omega := \sum_{i=1}^{n} (\partial\omega/\partial x_i) \cdot \theta_i.$$

Superpower ideals

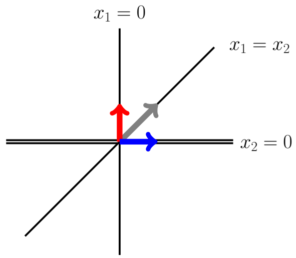
Our main object of study is the *superpower ideal*

$$I_{\mathcal{A}} := J_{\mathcal{A},1} + \{df : f \in J_{\mathcal{A},1}\}$$
$$:= \left(\lambda_L^{\rho_{\mathcal{A}}(L)+1}, d\lambda_L^{\rho_{\mathcal{A}}(L)+1} : L \subseteq \mathbb{C}^n \text{ a line} \right) \subseteq \Omega_n.$$

$I_{\mathcal{A}}$ can be thought of as the closure of the power ideal $J_{\mathcal{A},1}$ with respect to d.

An example

Consider the arrangement $\mathcal{A} = \{x_1 = 0, x_2 = 0, x_2 = 0, x_1 = x_2\} \subseteq \mathbb{R}^2$.



Line L	$x_1 = 0$	$x_2 = 0$	$x_2 = 0$	$x_1 = x_2$	Generator for $J_{\mathcal{A},1}$
$\mathbb{R} \cdot (1,0)$	$\not\subseteq$			$\not\subseteq$	x_1^{2+1}
$\mathbb{R} \cdot (0,1)$		$\not\subseteq$	$\not\subseteq$	$\not\subseteq$	x_2^{3+1}
$\mathbb{R} \cdot (1,1)$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$		$(x_1+x_2)^{3+1}$

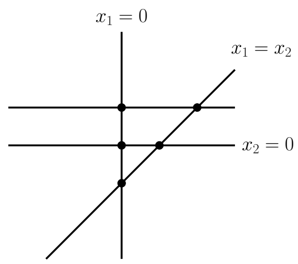
In fact, these terms generate the power ideal $J_{\mathcal{A},1}$:

$$J_{\mathcal{A},1} = (x_1^3, x_2^4, (x_1+x_2)^4) \subseteq \mathbb{R}[x_1, x_2]$$
$$I_{\mathcal{A}} = J_{\mathcal{A},1} + (x_1^2\theta_1, x_2^3\theta_2, (x_1+x_2)^3(\theta_1+\theta_2)) \subseteq \Omega_2.$$

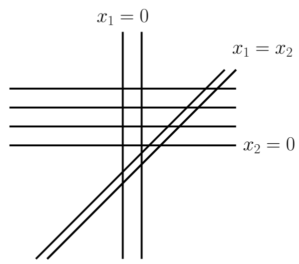
The Hilbert series of $\Omega_2/I_{\mathcal{A}}$ is

	q^0	q^1	q^2	q^3	q^4	row sums
t^0	1	2	3	3	1	10
t^1	2	4	5	3		14
t^2	1	2	2			5

where q tracks x degree and t tracks θ degree. The Theorem can be used to translate between the Tutte polynomial $T_{\mathcal{A}}(x,y) = x^2 + xy + y^2 + x + y$ and the Hilbert series. Corollary 1 explains the t^0 row, the t^2 row, and the rightmost diagonal, respectively.



$\mathcal{A}^{\text{gen}}$



$2\mathcal{A}^{\text{gen}}$

The (reversed) f-vector of $\mathcal{A}^{\text{gen}}$ gives the row sums of the Hilbert series (Corollary 2). The 29 regions in $2\mathcal{A}^{\text{gen}}$ correspond to the sum of the entire Hilbert series table (Corollary 3). Bounded versions of these quantities conjecturally govern $\Omega_n/I'_{\mathcal{A}}$.

Main theorem

Hilbert series and Tutte polynomial

Theorem. *For any rank r arrangement $\mathcal{A} \subseteq \mathbb{C}^n$ of m hyperplanes,*

$$\mathrm{Hilb}\left(\Omega_n/I_{\mathcal{A}}; q, t\right) = (1+t)^r q^{m-r} T_{\mathcal{A}}\left(\frac{1+q+t}{1+t}, \frac{1}{q}\right)$$

where q tracks x degree and t tracks θ degree.

This theorem is proved via an exact sequence analogous to deletion-restriction. We also give an explicit basis for the harmonic space $I_{\mathcal{A}}^{\perp}$ in terms of activities.

Corollaries and conjectures

External and central zonotopal algebras

Corollary 1. *Let $\mathcal{A} \subseteq \mathbb{C}^n$ be a rank r arrangement of m hyperplanes.*

- $\mathrm{Hilb}(\Omega_n/I_{\mathcal{A}}; q, 0) = \mathrm{Hilb}(\mathbb{C}[\boldsymbol{x}_n]/J_{\mathcal{A},1}; q)$.
- *The coefficient of t^r in $\mathrm{Hilb}(\Omega_n/I_{\mathcal{A}}; q, t)$ is $\mathrm{Hilb}(\mathbb{C}[\boldsymbol{x}_n]/J_{\mathcal{A},0}; q)$.*
- *The summand of maximal total degree in $\mathrm{Hilb}(\Omega_n/I_{\mathcal{A}}; q, t)$ is*

$$(-1)^r q^{m-r} t^r \chi_{\mathcal{A}}\left(-\frac{q}{t}\right).$$

where $\chi_{\mathcal{A}}$ is the characteristic polynomial of $\mathcal{A}$.

Real arrangements

Let $\mathcal{A}^{\text{gen}}$ be a generic affine deformation of a complexified real arrangement $\mathcal{A}$ and let f_i be the # of i-dimensional faces in the polyhedral complex induced by $\mathcal{A}^{\text{gen}}$.

Corollary 2. *For any complexified real arrangement $\mathcal{A}$,*

$$\mathrm{Hilb}(\mathcal{E}_{\mathcal{A}}; 1, t) = \sum_{i=0}^{n} f_{n-i}(\mathcal{A}^{\text{gen}}) \cdot t^i.$$

Corollary 3. *For any arrangement $\mathcal{A} \subseteq \mathbb{R}^n$,*

$$\mathrm{Hilb}(\Omega_n/I_{\mathcal{A}}; q, t)|_{(q,t)=(q^2,q)} = \mathrm{Hilb}(\mathbb{C}[\boldsymbol{x}_n]/J_{2\mathcal{A},1}; q).$$

Thus $\dim(\Omega_n/I_{\mathcal{A}})$ equals the number of regions in $2\mathcal{A}^{\text{gen}}$, the generic deformation of the arrangement in which each hyperplane in $\mathcal{A}$ appears twice.

A conjecture for the internal zonotopal algbebra

One could try to extend this work to $k = 0$ by defining

$$I'_{\mathcal{A}} := J_{\mathcal{A},0} + \{df : f \in J_{\mathcal{A},0}\} \subseteq \Omega_n.$$

One difficulty is that this case contains the $k = -1$ classical case, for which no basis of the Macaulay inverse is known.

Conjecture. *For any rank r arrangement $\mathcal{A} \subseteq \mathbb{C}^n$ of m hyperplanes,*

$$\mathrm{Hilb}\left(\Omega_n/I'_{\mathcal{A}}; q, t\right) = (1+t)^r q^{m-r} T_{\mathcal{A}}\left(\frac{1}{1+t}, \frac{1}{q}\right).$$