Degenerations, permutahedral subdivisions, and Coxeter elements

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Abstract

On the geometric side, we give degenerations of Lusztig varieties in G/B to unions of Richardson varieties and use this to obtain cohomological formulas. As a special case of this result, for each Coxeter element $c \in W$, we give a degeneration of the permutahedral variety in G/B to a union of toric Richardson varieties. On the combinatorial side, for each Coxeter element $c \in W$, we obtain a finest subdivision of the W-permutahedron into Bruhat interval polytopes. In types A and BC we additionally show that this subdivision is regular, providing connections to $\mathrm{Trop}^+\mathrm{Fl}_n$.

Combinatorial background

Notation. Simple transpositions are s_i, $\ell(w)$ is length, and $\leq$ is Bruhat order. A product uv is **length-additive** if $\ell(uv) = \ell(u) + \ell(v)$.

Definition. An element $c \in S_n$ is a **Coxeter element** if it has a reduced word where each s_i is used exactly once.

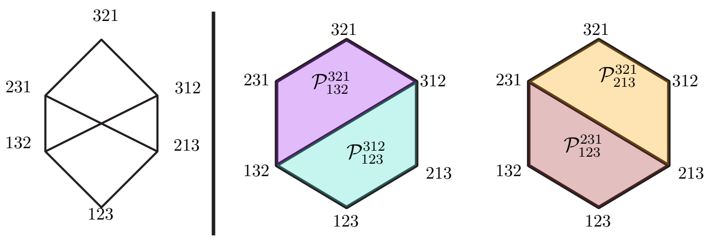
There are 2^{n-2} Coxeter elements in S_n. For $n = 4$, they are

$$s_1s_2s_3 \qquad s_1s_3s_2 = s_3s_1s_2 \qquad s_3s_2s_1 \qquad s_2s_3s_1 = s_2s_1s_3.$$

Definition ([1]). Let $\rho := (n, n-1, \ldots, 1) \in \mathbb{R}^n$. For $u \leq v \in S_n$, the **Bruhat interval polytope** (BIP) is

$$\mathcal{P}_u^v := \mathrm{conv}(z \cdot \rho : u \leq z \leq v).$$

Note $\mathcal{P}_e^{w_0}$ is the **permutahedron**. The right side of the figure below shows all Bruhat interval polytopes for $n = 3$.



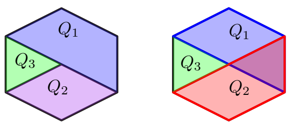
These definitions naturally extend to Weyl groups W. The polytope $\mathcal{P}_u^v$ is the moment polytope of $\mathcal{R}_u^v$.

Definition. For P, Q_i d-dimensional polytopes, a decomposition

$$P = \bigcup_{i \in I} Q_i$$

is a **subdivision** if for all $i, j \in I$, $Q_i \cap Q_j$ is a face of both Q_i and Q_j.

The figure above shows two subdivisions. The figure below shows two non-subdivisions.



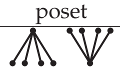
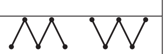
Combinatorial results

Theorem. *Let $c \in W$ be a Coxeter element. Then the decomposition*

$$\mathcal{P}_e^{w_0} = \bigcup_{\substack{u:uc \\ \text{length-add.}}} \mathcal{P}_u^{uc}$$

is a (finest) subdivision of the permutahedron into Bruhat interval polytopes. If W is type A or BC, this subdivision is regular.

Corollary. *Let $c \in S_n$ be a Coxeter element. The polytopes of the decomposition above are in bijection with linear extensions of the* inversion poset *of c.*

n	3	4	5	6	7	8	poset
$c = s_1 \ldots s_{n-1}$ or $c = s_{n-1} \ldots s_1$	2	6	24	120	720	5040	
$c =$(odds)(evens) or $c =$(evens)(odds)	2	5	16	61	272	1385	

Geometric background

Notation. Fix G a semisimple algebraic group, B Borel subgroup, B_- opposite Borel, $T = B \cap B_-$ maximal torus, W Weyl group.

For $x \in G$ and $w \in W$, the **Lusztig variety** is

$$\mathcal{Y}_w(x) := \{gB \in G/B : g^{-1}xg \in \overline{BwB}\}.$$

Theorem ([2]). *For $x \in G$ regular semisimple, $w \in W$, then $\mathcal{Y}_w(x)$ is smooth of pure dimension $\ell(w)$. The class $[\mathcal{Y}_w(x)] \in H^*(G/B)$ does not depend on x.*

For $w \in W$, the **Schubert variety** and **opposite Schubert variety** are

$$X^w := \overline{BwB/B} \qquad \text{and} \qquad X_w := \overline{B_-wB/B}.$$

For $u \leq v \in W$, the **Richardson variety** is the intersection

$$\mathcal{R}_u^v := X_u \cap X^v.$$

Geometric results

Theorem. *Let $x \in G$ be regular semisimple and let $w \in W$. There is a flat embedded degeneration*

$$\mathcal{Y}_w(x) \to \bigcup_{\substack{u:uw^{-1} \\ \text{length-add.}}} \mathcal{R}_u^{uw^{-1}} \quad \text{and we have} \quad [\mathcal{Y}_w(x)] = \sum_{\substack{u:uw^{-1} \\ \text{length-add.}}} [\mathcal{R}_u^{uw^{-1}}]$$

in $H^*(G/B)$.

If $c \in W$ is a Coxeter element, then $\mathcal{Y}_c(x) = \overline{T \cdot A}$ for some generic $A \in G/B$. That is, $\mathcal{Y}_c(x)$ is a *permutahedral variety*, a toric variety whose moment polytope is the W-permutahedron.

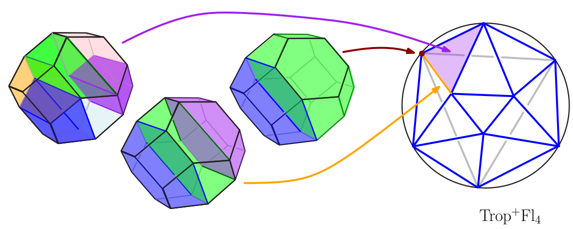
Corollary. *Let $c \in W$ be a Coxeter element, and $A \in G/B$ be generic. Then in $H^*(G/B)$*

$$[\overline{T \cdot A}] = \sum_{\substack{u:uc \\ \text{length-add.}}} [\mathcal{R}_u^{uc}].$$

In type A, all regular semisimple Hessenberg varieties are Lusztig varieties, for w a 312-avoiding permutation. So the theorem also gives degenerations and cohomological formulas for regular semisimple Hessenberg varieties.

Relation to $\mathrm{Trop}^+\mathrm{Fl}_n$

Theorem ([3, 4]). *In type A, (finest) regular subdivisions of $\mathcal{P}_e^{w_0}$ into BIPs are in bijection with (maximal) cones of $\mathrm{Trop}^+\mathrm{Fl}_n$.*



Corollary. *$\mathrm{Trop}^+\mathrm{Fl}_n$ has at least 2^{n-2} maximal cones. Finest regular subdivisions of $\mathcal{P}_e^{w_0}$ into BIPs do not always use the same number of polytopes.*

References

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