Jack Combinatorics of the Equivariant Edge Measure

Kyla Pohl and Ben Young
University of Oregon

University of Oregon Mathematics

The Jack Plancherel Measure

Definition: A *Young diagram* for a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ is the zero-indexed array of boxes in the plane with matrix coordinates

$$\{(i,j) | 0 \le i \le len(\lambda) - 1, 0 \le j \le \lambda_i - 1\}.$$

We identify a partitions with its Young diagram.

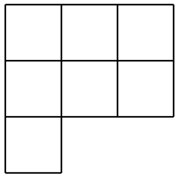


Figure 1. The diagram for the partition $\lambda = (3,3,1)$.

Definition: For a partition λ define the *upper and lower hook lengths* as

$$h_\lambda^*(i,j) = u \cdot \ell(\square) + v \cdot (a(\square) + 1)$$
$$h_*^\lambda(i,j) = u \cdot (\ell(\square) + 1) + v \cdot a(\square)$$

respectively, where $a(\square)$ and $\ell(\square)$ are the arm and leg lengths of $\square$ in λ.

$3v+u$ $2v+2u$	$2v+u$ $v+2u$	v u
$2v$ $v+u$	v u	

Figure 2. The upper and lower hook lengths of each box in $\mu = (3,2)$.

Definition [(1)]: The *Jack Plancherel measure* is a probability measure on partitions of an integer n, defined by

$$w_{Jack}(\lambda) = \frac{1}{\prod_{(i,j)\in\lambda} h_\lambda^*(i,j) h_*^\lambda(i,j)}.$$

Example: The product of all the entries in the boxes in the Figure 2 is $\frac{1}{w_{Jack}((3,2))}$.

Acknowledgements

The authors would like to thank Golnaz Bahrami, Houcine Ben Dali, Cesar Cuenca, Maciej Dołęga, Martijn Kool, Dan Romik, Chris Sinclair, Piotr Śnaidy, and Kayla Wright for helpful conversations. The presenter is supported by NSF grant DMS-2039316.

More question about this poster? Message me on Discord!

Abstract

We study the equivariant edge measure: a measure on partitions which arises implicitly in the edge term in the localization computation of the Donaldson--Thomas invariants of a toric threefold. We combinatorially show that the equivariant edge measure is, up to choices of convention, equal to the Jack Plancherel measure.

Main Result

Theorem: The Jack Plancherel measure of a partition λ is the same as the equivariant edge measure of λ up to a sign, i.e.

$$\boldsymbol{w_{Jack}(\lambda) = -w_{MNOP}(\lambda).}$$

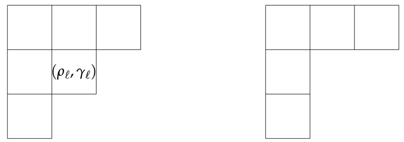
Our main contribution is not the theorem itself, which is known, at least implicitly, to geometers who study Hilbert schemes, but rather

- a direct combinatorial proof of the purely combinatorial theorem statement, which does not require any geometry, and
- the observation that the *corner polynomial* (described below) arises naturally in the induction step, and that a similar construction is likely relevant in the much harder *vertex measure* (see Future Work).

We show this via a comparison of ratios of the two objects under consideration. More specifically, we consider the ratios

$$\frac{w_{Jack}(\lambda)}{w_{Jack}(\mu)} \text{ and } \frac{w_{MNOP}(\lambda)}{w_{MNOP}(\mu)}$$

where $\mu \subseteq \lambda$ and $|\lambda| = |\mu| + 1$. In other words, μ is λ with one corner missing.



(a) $\lambda = (3,2,1)$ (b) $\mu = (3,1,1)$

Figure 4. The Young diagram $\mu = (3,1,1)$ is obtained by removing the corner (ρ_ℓ, γ_ℓ) from $\lambda = (3,2,1)$.

Our result is proven by showing that the two ratios described are equal for all pairs of partitions as above with smaller partition of size at least one. Routine induction then yields our result. Significant cancellation occurs in both ratios. In $\frac{w_{MNOP}(\lambda)}{w_{MNOP}(\mu)}$ an expression which we label the *corner polynomial* also appears.

1			−1	
		−1	1	
−1		1		

Figure 5. Inside every cell in λ = (3, 2) is the coefficient of its contribution to C.

Definition: The *corner polynomial* of a partition λ is

$$C = C(\lambda) = 1 + \sum_{(i,j)\text{ inside corner of }\lambda} r^{i+1}s^{j+1} - \sum_{(i,j)\text{ ouside corner of }\lambda} r^i s^j.$$

Future Work

Our work so far serves as a warm-up exercise for the three dimensional version of this problem. In other words, we aim to combinatorially describe the equivariant *vertex* measure. We expect to use similar techniques in our work on that problem. Indeed, an analogue of the corner polynomial, this time involving saddle points as well, appears.

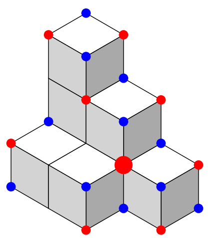


Figure 6. (right) Each type of corner/saddle contributes a predetermined amount to the 3D corner polynomial.

The Equivariant Edge Measure

Definition: Given an index set A and a Laurent polynomial $G = \sum_{(i,j)\in A} c_{i,j} r^i s^j$ in the variables r and s with no constant term, we define the *swap operation* as follows:

$$swap(G) = swap\left(\sum_{(i,j)\in A} c_{i,j} r^i s^j\right) = \prod_{(i,j)\in A} (iu - jv)^{c_{i,j}}.$$

Essentially, the roles of addition and multiplication have been swapped. Note that the variables r and s have been changed to u and v.

Definition[3]: Given a partition λ, define generating functions

$$Q(\lambda) = \sum_{(i,j)\in\lambda} r^i s^j$$

and

$$\overline{Q}(\lambda) = \sum_{(i,j)\in\lambda} r^i s^j$$

where the sums are taken over the coordinates of all cells in λ. From these, define

$$F(\lambda) = -Q(\lambda) - \frac{\overline{Q}(\lambda)}{rs} + \frac{Q(\lambda)\overline{Q}(\lambda)(1-r)(1-s)}{rs}.$$

Note that Q and $\overline{Q}$ each assign a monomial to every square in the Young diagram λ which describes that cell's zero-indexed matrix coordinates.

r^0s^0	r^0s^1	r^0s^2
r^1s^0	r^1s^1	r^1s^2
r^2s^0		

Figure 3. The monomial contributed to Q by each cell in $\mu = (3,3,1)$.

Definition: The *equivariant edge measure*, w_{MNOP}, is the swap operation applied to $F(\lambda)$.

References

[1] A. Borodin and G. Olshanski. "Z-measures on partitions and their scaling limits". European Journal of Combinatorics 26.6 (2005), pp. 795–834.

[2] R. P. Stanley. "Some combinatorial properties of Jack symmetric functions". Advances in Mathematics 77.1 (1989), pp. 76–115.

[3] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. "Gromov–Witten theory and Donaldson–Thomas theory, i". Compositio Mathematica 142.5 (2006), pp. 1263–1285.

[4] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. "Gromov–Witten theory and Donaldson–Thomas theory, ii". Compositio Mathematica 142.5 (2006), pp. 1286–1304.