Fundamental polytopes and associated hyperplane arrangement

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Project goal

To analyse fundamental polytopes of strict and generic metrics and check compatibility with other classifications, such as via tight spans (and by extension split-decomposability). Furthermore, to connect the results to previous research and analyse the new hyperplane arrangement.

Metric spaces

An *n-metric* is an $n \times n$ matrix d with entries d_{ij} for $1 \leq i,j \leq n$ satisfying
1. $d_{ii} = 0$ for all $1 \leq i \leq n$,
2. $d_{ij} > 0$ for all $1 \leq i \neq j \leq n$,
3. $d_{ij} = d_{ji}$ for all $1 \leq i < j \leq n$ and
4. $d_{ij} + d_{jk} \geq d_{ik}$ for all $1 \leq i,j,k \leq n$.

The metric d is called strict if $d_{ij} + d_{jk} > d_{ik}$ for all $j \in [n] \setminus \{i,k\}$.

Fundamental polytopes

The fundamental polytope or Kantorovich-Rubinstein-Wasserstein polytope of an n-metric d is a polytope in $\mathbb{R}^n$ defined as the convex hull:
$$\mathrm{KRW}(d) = \mathrm{conv}\left\{\frac{e_i - e_j}{d_{ij}} \;\middle|\; 1 \leq i < j \leq n\right\}.$$
A strict metric d is called generic if $\mathrm{KRW}(d)$ is simplicial.
Define
$$e_{ij} := \frac{e_i - e_j}{d_{ij}} \text{ for } i \neq j.$$
The Lipschitz polytope of d is given as an intersection of half-spaces:
$$\mathrm{LIP}(d) := \{x \in \mathbb{R}^n \mid \textstyle\sum_i x_i = 0, x_i - x_j \leq d_{ij} \text{ for all } 1 \leq i,j \leq n\}.$$
$\mathrm{LIP}(d)$ and $\mathrm{KRW}(d)$ are dual polytopes.

If d is the n-metric with $d_{ij} = 1$ for all $i \neq j$, then $\mathrm{KRW}(d)$ is the type A_n root polytope.

Tight span of a metric

Given an metric d on n points, consider the (unbounded) polyhedron
$$P(d) := \{x \in \mathbb{R}^n \mid x_i + x_j \geq d_{ij} \text{ for all } i,j \in [n]\}.$$
The tight span $E(d)$ of d is the set of coordinate-wise minimal elements of $P(d)$.

The metric cone and metric fan

Consider the vector space $\mathbb{R}^{\binom{n}{2}}$ with coordinates x_{ij} indexed by pairs of elements of $[n]$. The metric cone on n elements is the subset $\mathbf{M}_n \subseteq \mathbb{R}^{\binom{n}{2}}$ defined by
$$x_{ij} > 0, \quad x_{ij} + x_{jk} \geq x_{ik}, \text{ for all pairwise distinct } i,j,k \in [n].$$
The metric fan MF_n is the secondary fan of the second hypersimplex
$$\Delta(n,2) := \mathrm{conv}\{e_i + e_j : 1 \leq i < j \leq n\} \subset \mathbb{R}^n.$$

Split-decomposable metrics

Let X be a finite set. A *split* σ of X is a bipartition of X. We write splits as pairs: $\sigma = A|B$.
We can associate a function to a split as follows:
$$\delta_\sigma(i,j) = \begin{cases} 0 & i \sim_\sigma j \\ 1 & \text{otherwise} \end{cases}.$$
A metric d is called split-decomposable if it can be expressed as
$$d = \Sigma \lambda_{A,B} \cdot \delta_{A,B}.$$

Combinatorial structure

To each point e_{ij} associate an oriented edge. Given a face α of $KRW(d)$ define a graph $\tilde{D}(\alpha)$ on the vertex set $[n]$ and edges corresponding to e_{ij} lying on α. Denote by $D(\alpha)$ the unoriented version of $\tilde{D}(\alpha)$.

Given a dual pair of polytopes $KRW(d), \mathrm{LIP}(d)$, an oriented (resp. unoriented) graph G is called admissible, if there exists a face α, such that all edges of G belong to $\tilde{D}(\alpha)$ (resp. $D(\alpha)$).

The collection of graphs of the form $D(\alpha)$ is called the combinatorial structure of the dual pair.
Two metrics are Lipschitz combinatorially equivalent if the combinatorial structures of the respective polytopes coincide.

Tool for determining the combinatorial structure [GP]

The following are equivalent for a metric d on a set X and a directed graph G:
(i) G is admissible.
(ii) For any array of oriented edges $(x_i, y_i), 1 \leq i \leq k$ with all x_i and all y_i distinct:
$$\sum_{i=1}^{k} d(x_i, y_i) \leq \sum_{i=1}^{k} d(x_i, y_{i+1}),$$
where $y_{k+1} = y_1$.

Furthermore, a strict metric is generic if and only if for any $2k$ distinct points $x_1, .., x_k, y_1, .., y_k$ in X the minimum of the terms
$$\sum_{i=1}^{k} d(x_i, y_{\pi(i)})$$
is attained by a unique permutation π.

Meaning per $2k$ points, there is only one maximal set of admissible edges.

Main objective (Vershik, 2015)

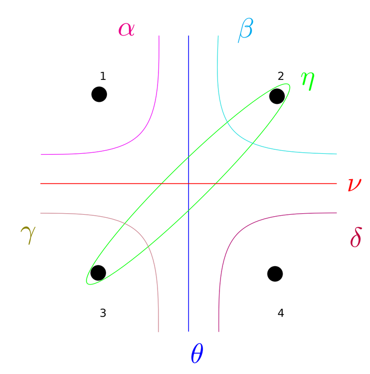
Combinatorial classification of metrics spaces by means of the fundamental polytopes.

Classification-related Research

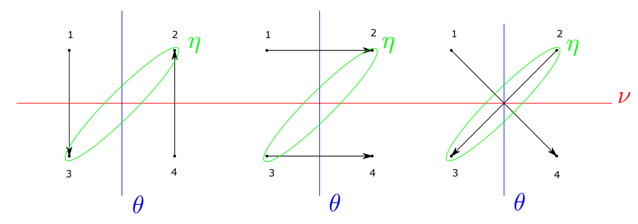
'92 Classification based on split-decomposability and tight span. (Bandelt, Dress)
'04 Classification of six-point metrics by use of polyhedral subdivisions of the metric fan. (Sturmfels, Yu)
'17 Bounds for number of possible different f-vectors, given n and computation of face numbers in the generic case (in particular, the f-vector is unique in that case). (Gordon, Petrov)
'20 Computation of the f-vector for tree-like metric spaces. (Delucchi, Hoessly)

Example: generic metrics on four points

All metrics on four points are split-decomposable.
Splits:



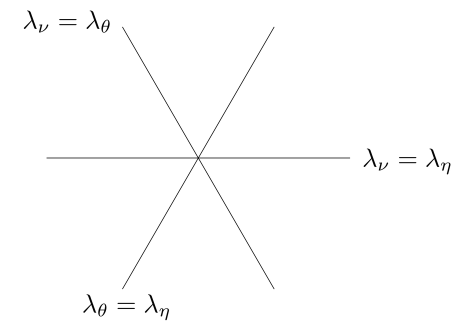
Possible cycles to check:



Computations:
$$d(1,3) + d(2,4) \leq d(1,2) + d(3,4)$$
$$\Leftrightarrow \alpha + \gamma + \nu + \eta + \beta + \delta + \nu + \eta \leq \alpha + \beta + \eta + \theta + \gamma + \delta + \eta + \theta$$
$$\Leftrightarrow \nu \leq \theta$$

Analogue:
$$d(1,2) + d(3,4) \leq d(1,4) + d(2,3) \Leftrightarrow \eta \leq \nu$$
$$d(1,4) + d(2,3) \leq d(4,2) + d(3,1) \Leftrightarrow \theta \leq \eta$$

Cycle arrangement on four points:



The Wasserstein arrangement

We define a hyperplane arrangement in $\mathbb{R}^{\binom{n}{2}}$ as follows. Given $k > 0$ and k-tuples $a, b \in [n]^k$ define
$$H_{a,b} := \left\{x \in \mathbb{R}^{\binom{n}{2}} \,\middle|\, \sum_{i=1,\dots,k} x_{a_i b_i} = \sum_{i=1,\dots,k} x_{a_i b_{i+1}}\right\}, \quad \text{where } b_{k+1} = b_1.$$
The 'positive side' of $H_{a,b}$ is
$$H^+_{a,b} := \left\{x \in \mathbb{R}^{\binom{n}{2}} \,\middle|\, \sum_{i=1,\dots,k} x_{a_i b_i} \leq \sum_{i=1,\dots,k} x_{a_i b_{i+1}}\right\}$$
The Wasserstein arrangement is then the set of hyperplanes
$$\mathcal{W}_n := \left\{H_{a,b} \,\middle|\, \begin{array}{l} 0 < k \leq n, a, b \in [n]^k \\ a_1, \dots, a_k, b_1, \dots, b_k \text{ mutually distinct} \end{array}\right\}.$$
The hyperplane $H_{a,b}$ depends only on the cycle C of the complete graph K_n determined by the sequence of vertices $a_1, b_1, a_2, \dots, a_k, b_k$.
Remark: We need only look at these types of graphs because we consider the hyperplane in the metric cone.

Relationship between fans

Let $n \geq 2$ and let $\mathcal{W}^4_n$ denote the subset of $\mathcal{W}_n$ consisting of all hyperplanes of the form $H_{(i,j),(k,l)}$ for distinct $i,j,k,l \in [n]$.
1. For every $H \in \mathcal{W}^4_n$, the set $H \cap \mathbf{M}_n$ is a union of cones of the type fan $\mathscr{T}_n$.
2. For $n \leq 5$, the Wasserstein fan $\mathscr{F}_n$ agrees with the type fan $\mathscr{T}_n$.
3. For $n \geq 6$, the Wasserstein fan $\mathscr{F}_n$ is a strict refinement of the type fan $\mathscr{T}_n$.

Some computational data (Julia and TOPCOM [R])

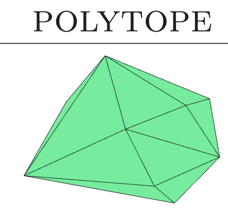
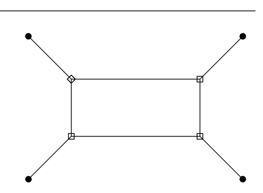
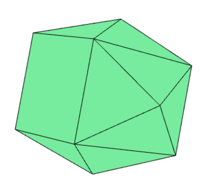
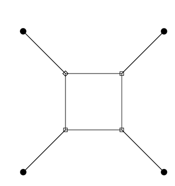
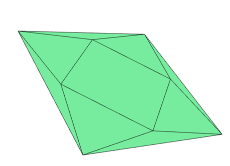
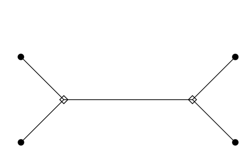
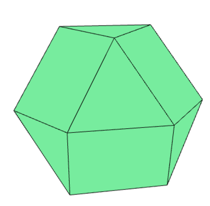
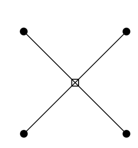
The table only considers types of generic metrics. The chambers of the arrangement are not in 1-to-1 correspondence to the labeled types.

n	Unlabeled	Labeled	# Chambers
3	1	1	1
4	1	6	6
5	12	882	882
6	25,224[1]	17,695,320[1]	6,677,863,200

Another way of computing this for generic metrics is to enumerate the number of possible triangulations of the root polytope.

[1] Computed by J. Rambau via triangulations of the root polytope

Types of four point metrics

Nr.	Case	f-vector	Polytope	Tight span
1	$d_{12\|34} > d_{13\|24} > d_{14\|23}$	(12 30 20)		
2	$d_{12\|34} > d_{13\|24} = d_{14\|23}$	(12 28 18)		
3	$d_{12\|34} = d_{13\|24} > d_{14\|23}$	(12 28 18)		
4	$d_{12\|34} = d_{13\|24} = d_{14\|23}$	(12 24 14)		

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