Extending the Science Fiction and the Loehr–Warrington Formula

Donghyun Kim* and Jaeseong Oh+

Seoul National University* and Korea Institute for Advanced Study+

Garsia–Haiman modules

For a partition μ of N, the *generalized Vandermonde determinant* is defined by

$$\Delta_\mu := \det\left(x_k^{i-1} y_k^{j-1}\right)_{\substack{1\le k\le N,\\ (i,j)\in\mu}}$$

and the *Garsia–Haiman module* [3] is

$$R_\mu = \operatorname{span}\{\partial_x^I \partial_y^J \Delta_\mu\} \subseteq \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n],$$

that is, the span of all partial derivatives of Δ_μ.
Haiman [6] used the geometry of the Hilbert scheme to prove that

$$\dim R_\mu = N!, \tag{1}$$

$$\operatorname{Frob}(R_\mu) = \widetilde{H}_\mu[X;q,t], \tag{2}$$

where $\widetilde{H}_\mu$ is the *(modified) Macdonald polynomial*. The first result is known as the $n!$ *theorem*, and the second confirms the *Macdonald positivity conjecture*.

Science fiction conjecture

Let μ be a partition of N with n (removable) corners. For $1\le i\le n$, let $\mu^{(i)}$ denote the partition obtained from μ by removing its i-th corner. The science fiction conjecture of Bergeron and Garsia [1] then implies the following two assertions, which are analogous to (1) and (2), respectively:

$$\dim\left(\bigcap_{i=1}^n R_{\mu^{(i)}}\right) = \frac{N!}{n}, \tag{3}$$

$$\operatorname{Frob}\left(\bigcap_{i=1}^n R_{\mu^{(i)}}\right) = \sum_{i=1}^n \left(\prod_{j\ne i}\frac{T_{\mu^{(j)}}}{T_{\mu^{(j)}}-T_{\mu^{(i)}}}\right)\widetilde{H}_{\mu^{(i)}}[X;q,t]. \tag{4}$$

The symmetric function on the right-hand side of (4) is called the *Macdonald intersection polynomial*, and is denoted by $\mathrm{I}_{\mu^{(1)},\dots,\mu^{(n)}}$.

Macdonald piece polynomials

Let μ be a partition of N with n corners $\{c_1,\dots,c_n\}$, and let $1\le k\le n$. For any subset $S\in\binom{[n]}{k}$, let μ^S denote the partition obtained from μ by removing the corners $\{c_i : i\in S\}$. Let $R_{n,k} := ((n-k)^k)$ be the rectangular partition, and let λ be a partition contained in $R_{n,k}$. We define the *Macdonald piece polynomial* by

$$\mathrm{I}_{\mu,\lambda,k}[X;q,t] := (-1)^{|\lambda|}\sum_{S\in\binom{[n]}{k}} \frac{s_\lambda[z_S]\prod_{j\in S^c} z_j}{\prod_{\substack{i\in S\\ j\in S^c}}(z_j - z_i)}\widetilde{H}_{\mu^S}[X]. \tag{5}$$

We further define

$$\widetilde{H}^{\lambda}_{\mu^S} := \Phi\left(\sum_{\nu\subseteq R(n,k)} \pi_{\lambda,S}\frac{s_{\tilde{\nu}}[z_{S^c}]}{\prod_{j\in S^c} z_j}\mathrm{I}_{\mu,\nu,k}\right), \tag{6}$$

where $\pi_{\lambda,S}$ denotes a composition of certain Demazure-like operators, and Φ is the map sending each z_i to $T_{\mu^{(i)}}/T_\mu$.

Conjecture. For certain set $S(\lambda)$ associated to λ, have

$$\widetilde{H}^{\lambda}_{\mu^S} = \operatorname{Frob}\Big(\bigcap_{S'\in S(\lambda)} R_{\mu^{S'}}\Big). \tag{7}$$

Loehr–Warrington conjecture

The *Nabla operator* ∇ is the diagonal operator with respect to the Macdonald basis, defined by

$$\nabla\widetilde{H}_\mu = T_\mu \widetilde{H}_\mu.$$

The Loehr–Warrington conjecture [8] was very recently proved in [2], providing a combinatorial formula for ∇s_λ in terms of *labeled nested Dyck paths* as

$$\nabla s_\lambda = (-1)^{\operatorname{adj}(\lambda)}\sum_{(G,R)\in \mathrm{LNDP}_\lambda} q^{\operatorname{dinv}(G,R)} t^{\operatorname{area}(G,R)} x_R. \tag{8}$$

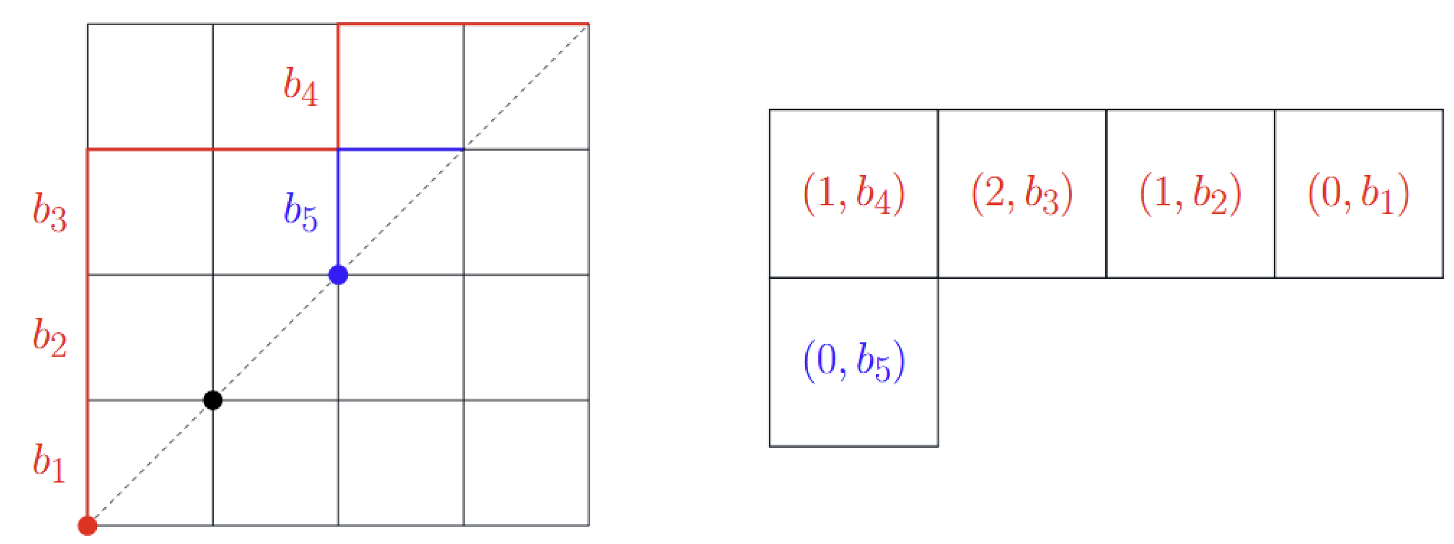

Fig. 1: A labeled nested Dyck path in $\mathrm{LNDP}_{(3,2)}$ and a $\mathbf{P}$-tableau in $\mathcal{T}(3,2)$.

Reformulation of the Loehr–Warrington formula

Consider the poset $\mathbf{P}$ on $\mathbb{Z}_{\ge 0}\times\mathbb{Z}_{\ge 1}$ defined as follows. For $(a,b),(c,d)\in\mathbb{Z}_{\ge 0}\times\mathbb{Z}_{\ge 1}$, we write $(a,b)\prec_{\mathbf{P}}(c,d)$ if and only if either $a+1<c$, or $a+1=c$ and $b\ge d$. Otherwise, we write $(a,b)\not\prec_{\mathbf{P}}(c,d)$.
Define the set of $\mathbf{P}$-*tableaux* $\mathcal{T}(\lambda)$ to consist of fillings T of the diagram $D(\lambda)$ satisfying the following conditions:

- $T(i+1,j)\succ_{\mathbf{P}} T(i,j)$ and $T(i,j-1)\not\succ_{\mathbf{P}} T(i,j)$,
- for each $j>k-s$ with $j\in\operatorname{piv}(\lambda)$, we have $T(\operatorname{bot}(\lambda)_j,j)_1>0$,
- for each $j>k-s$ with $j\notin\operatorname{piv}(\lambda)$, we have $T(\operatorname{bot}(\lambda)_j,j)_1=0$.

Then the Loehr–Warrington formula (the right-hand side of (8)) can be reformulated as

$$\mathrm{LW}_\lambda := q^{\operatorname{adj}(\lambda)}\sum_{T\in\mathcal{T}(\lambda)} q^{\operatorname{dinv}(T)} t^{\operatorname{area}(T)} x^T. \tag{9}$$

For example, the labeled nested Dyck path on the left in **Fig. 1** corresponds to the $\mathbf{P}$-tableau on the right in **Fig. 2**.
Using this $\mathbf{P}$-tableaux description (9), we obtain a *Jacobi–Trudi*-type formula for LW_λ:

$$q^{-\operatorname{adj}(\lambda)}\,\mathrm{LW}_\lambda = \det(W(\lambda))\cdot 1\Big|_{y_{i,j}=t^i x_j}, \tag{10}$$

where $W(\lambda)$ is a certain matrix of $\mathfrak{h}$-operators that generates each column of the $\mathbf{P}$-tableaux. In our running example, we have

$$W(3,2) = \begin{pmatrix} \bar{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_3 & \bar{\mathfrak{h}}_4 & \hat{\mathfrak{h}}_4\\ \bar{\mathfrak{h}}_1 & \hat{\mathfrak{h}}_1 & \hat{\mathfrak{h}}_2 & \bar{\mathfrak{h}}_3 & \hat{\mathfrak{h}}_3\\ \bar{\mathfrak{h}}_0 & \hat{\mathfrak{h}}_0 & \hat{\mathfrak{h}}_1 & \bar{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_2\\ 0 & 0 & \hat{\mathfrak{h}}_0 & \bar{\mathfrak{h}}_1 & \hat{\mathfrak{h}}_1\\ 0 & 0 & 0 & \bar{\mathfrak{h}}_0 & \hat{\mathfrak{h}}_0 \end{pmatrix}.$$

Main Results

Theorem. [7]

$$\frac{1}{T_{\mu^{[n]}}} e^{\perp}_{|\mu|-|\tilde{\lambda}|-k}\,\mathrm{I}_{\mu,\lambda,k} = \nabla s_{\tilde{\lambda}}. \tag{11}$$

$$\frac{1}{T_{\mu^{[n]}}} e^{\perp}_{|\mu|-|\tilde{\lambda}|-k}\,\mathrm{I}_{\mu,\lambda,k} = (-1)^{\operatorname{adj}(\tilde{\lambda})}\,\mathrm{LW}_{\tilde{\lambda}}. \tag{12}$$

Corollary. Loehr–Warrington conjecture

Main tools

For the first part, we need two lemmas: Garsia–Haiman–Tesler's plethystic formula [4] and one derived from Jacobi's bi-alternant formula for Schur functions.

Lemma. For $\mu\vdash n$,

$$e^{\perp}_{n-m}\widetilde{H}_\mu = (qt)^m T_\mu \sum_{\lambda\vdash m}\operatorname{rev}\left(\Pi'_{\widetilde{H}_\lambda}[D_\mu;q,t]\right)\frac{T_\lambda\widetilde{H}_\lambda}{\tilde{h}_\lambda\tilde{h}'_\lambda}. \tag{13}$$

Lemma. For partitions $\lambda\subseteq R(n,k)$ and μ with $|\mu|\le k(n-k)-|\lambda|$,

$$\sum_{S\in\binom{[n]}{k}}\frac{s_\lambda[z_S]\,s_\mu[z_{S^c}]}{\prod_{i\in S,\, j\in S^c}(z_j-z_i)} = (-1)^{|\lambda|}\delta_{\tilde{\lambda},\mu}. \tag{14}$$

Our new technique for the second part is shown in the following example with switching determinants.

$$W(3,2) = \begin{pmatrix} \bar{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_3 & \bar{\mathfrak{h}}_4 & \hat{\mathfrak{h}}_4\\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \xrightarrow[\text{Step1}]{} \begin{pmatrix} \bar{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_3 & \bar{\mathfrak{h}}_4 & \hat{\mathfrak{h}}_4\\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \xrightarrow[\text{Step2}]{} \begin{pmatrix} \mathfrak{h}_2 & \hat{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_3 & \bar{\mathfrak{h}}_4 & \hat{\mathfrak{h}}_4\\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = W^{(1)},$$

$$W^{(1)} = \begin{pmatrix} \mathfrak{h}_2 & \hat{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_3 & \bar{\mathfrak{h}}_4 & \hat{\mathfrak{h}}_4\\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \xrightarrow[\text{Step1}]{} \begin{pmatrix} \mathfrak{h}_2 & \bar{\mathfrak{h}}_4 & \hat{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_3 & \hat{\mathfrak{h}}_4\\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \xrightarrow[\text{Step2}]{} \begin{pmatrix} \mathfrak{h}_2 & \mathfrak{h}_4 & \hat{\mathfrak{h}}_2 & \hat{\mathfrak{h}}_3 & \hat{\mathfrak{h}}_4\\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} = W^{(2)}.$$

We then obtain

$$\mathrm{LW}_\lambda = (-1)^{\operatorname{adj}(\lambda)}\det(W^{(s)})\cdot 1\Big|_{y_{i,j}=t^i x_j},$$

which matches the combinatorial formula of Haglund–Haiman–Loehr [5].

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