Conjectures on the Reduced Kronecker Coefficients

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The Okounkov conjecture

Recall that the **Littlewood–Richardson coefficients** $c^{\nu}_{\lambda\mu}$ are the structure constants of the tensor product of irreducible polynomial representations of general linear group $GL_n(\mathbb{C})$:

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} c^{\nu}_{\lambda\mu} V(\nu),$$

where λ, μ, and ν are partitions with lengths less than or equal to n. Okounkov made the following

Okounkov Conjecture

The function

$$(\lambda, \mu, \nu) \to \log c^{\nu}_{\lambda\mu}$$

is concave. That is, suppose $(\lambda_i, \mu_i, \nu_i), i = 1, 2, 3$, are partitions such that

$$(\lambda_2, \mu_2, \nu_2) = \frac{1}{2}(\lambda_1, \mu_1, \nu_1) + \frac{1}{2}(\lambda_3, \mu_3, \nu_3),$$

then we have

$$(c^{\nu_2}_{\lambda_2\mu_2})^2 \geq c^{\nu_1}_{\lambda_1\mu_1} c^{\nu_3}_{\lambda_3\mu_3}.$$

Okounkov's conjecture holds in the classical limit, but it is false in general. However, an interesting implication of Okounkov conjecture is true. Okounkov conjecture would have implied that for all ν,

$$c^{\nu}_{\frac{\lambda+\mu}{2}\frac{\lambda+\mu}{2}} \geq c^{\nu}_{\lambda\mu},$$

provided $\frac{\lambda+\mu}{2}$ is an integral weight (a.k.a., a partition). It is equivalent to the inclusion

$$V(\lambda) \otimes V(\mu) \subset V\left(\frac{\lambda+\mu}{2}\right)^{\otimes 2},$$

which can be interpreted as saying that the representation valued function

$$V : \lambda \mapsto V(\lambda)$$

is concave with respect to the natural ordering and tensor product and is equivalent to the following

Lam–Postnikov–Pylyavskyy Theorem

For two partitions λ and μ, suppose $\lambda + \mu$ has only even parts and let s_λ, s_μ, and $s_{\frac{\lambda+\mu}{2}}$ be the corresponding Schur polynomials, then $s^2_{\frac{\lambda+\mu}{2}} - s_\lambda s_\mu$ is a non-negative linear combination of Schur polynomials.

Reduced Kronecker coefficients

Recall that the Kronecker coefficients $g^{\nu}_{\lambda\mu}$ are the structure constants of the tensor product (Kronecker product) of irreducible representations of the symmetric group S_d:

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} g^{\nu}_{\lambda\mu} V_\nu,$$

where λ, μ and ν are partitions of d. They were introduced by Murnaghan in 1938 and they play an important role algebraic combinatorics and geometric complexity theory.

For λ a partition and $d \geq |\lambda| + \lambda_1$, the "padded" partition $\lambda[d]$ is defined as $(d - |\lambda|, \lambda)$, which is a partition of size d with a "very long top row".

It was noticed by Murnaghan that the sequence $\left\{g^{\nu[d]}_{\lambda[d],\mu[d]}\right\}_{d>>0}$ stabilizes and the stable value of the sequence was called the **reduced (or stable) Kronecker coefficient** $\bar{g}^{\nu}_{\lambda\mu}$ associated with the triple (λ, μ, ν). Given λ and μ, only finitely many $\bar{g}^{\nu}_{\lambda\mu}$ are nonzero. Moreover, $\bar{g}^{\nu}_{\lambda\mu} = 0$ unless the Murnaghan–Littlewood inequality holds:

$$|\nu| \leq |\mu| + |\lambda|, |\mu| \leq |\lambda| + |\nu|, |\lambda| \leq |\mu| + |\nu|.$$

In contrast to Kronecker coefficients, reduced Kronecker coefficients are defined for any triple of partitions (not necessarily of the same size) and in general, there is no relationship between λ, μ, and ν. However, surprisingly, when $|\nu| = |\lambda| + |\mu|$, the reduced Kronecker coefficient $\bar{g}^{\nu}_{\lambda\mu}$ recovers the Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$.

Murnaghan–Littlewood theorem

If $|\nu| = |\lambda| + |\mu|$, then the reduced Kronecker coefficient $\bar{g}^{\nu}_{\lambda\mu}$ is equal to the the Littlewood–Richardson coefficients $c^{\nu}_{\lambda\mu}$: $\bar{g}^{\nu}_{\lambda\mu} = c^{\nu}_{\lambda\mu}$.

Additionally, every Kronecker coefficient is equal to an explicit reduced Kronecker coefficient of not much larger partitions.

Now we formulate a series of conjectures on the stable tensor product of irreducible representations of symmetric groups, which are closely related to the reduced Kronecker coefficients. These conjectures are certain generalizations of the Okounkov conjecture and the Schur log-concavity theorem of Lam–Postnikov–Pylyavskyy.

Conjectures and evidence

One Conjecture

The reduced Kronecker coefficients satisfy the following inequality: given λ and μ, then for all ν, we have

$$\bar{g}^{\nu}_{\frac{\lambda+\mu}{2}\frac{\lambda+\mu}{2}} \geq \bar{g}^{\nu}_{\lambda\mu},$$

provided $\frac{\lambda+\mu}{2}$ is still a partition.

We tested the above statement for all partitions λ and μ with at most 11 boxes. Using existing combinatorial interpretation of Kronecker coefficients with two two-row partitions, we prove the following

Proposition

The above conjecture holds when partitions λ and μ are both one part. Actually, we have the following stronger inequalities: for all partition ν,

$$\bar{g}^{\nu}_{(j)(k)} \geq \bar{g}^{\nu}_{(i)(l)},$$

whenever $i < j \leq k < l$ with $j + k = i + l$.

We can restate the above conjecture by using the stable representation category $\mathrm{Rep}(S_\infty)$ of symmetric group in [1]: the representation valued function

$$V : \quad \mathcal{P} \to \mathrm{Rep}\,(S_\infty)$$
$$\lambda \longmapsto V_{\lambda[\infty]}$$

is concave with respect to the natural ordering and tensor products of representations. That is,

$$V^{\otimes 2}_{\frac{\lambda+\mu}{2}[\infty]} \geq V_{\lambda[\infty]} \otimes V_{\mu[\infty]} \tag{1}$$

in the Grothendieck ring $\mathrm{K}(\mathrm{Rep}(S_\infty))$. We prove the following log-concavity property of the dimensions of representations in (1).

Proposition

$$\left(\dim V_{\frac{\lambda+\mu}{2}[d]}\right)^2 \geq \dim V_{\lambda[d]} \times \dim V_{\mu[d]}$$

for $d \geq \max\{|\lambda| + \lambda_1, |\mu| + \mu_1\}$. In another form,

$$\left(f^{\frac{\lambda+\mu}{2}[d]}\right)^2 \geq f^{\lambda[d]} \times f^{\mu[d]},$$

where f^λ denotes the number of standard Young tableaux of shape λ.

For two partitions λ and μ, let $\lambda \cup \mu = (\nu_1, \nu_2, \nu_3, \ldots)$ be the partition obtained by rearranging all parts of λ and μ in the weakly decreasing order. Let $\mathrm{sort}_1(\lambda, \mu) := (\nu_1, \nu_3, \nu_5, \ldots)$ and $\mathrm{sort}_2(\lambda, \mu) := (\nu_2, \nu_4, \nu_6, \ldots)$. Then, we have

Another conjecture

For two partitions λ and μ, we have

$$V_{\mathrm{sort}_1(\lambda,\mu)[\infty]} \otimes V_{\mathrm{sort}_2(\lambda,\mu)[\infty]} \geq V_{\lambda[\infty]} \otimes V_{\mu[\infty]}$$

in the Grothendieck ring $\mathrm{K}(\mathrm{Rep}(S_\infty))$.

Using the existing combinatorial interpretation of Kronecker coefficients with two hook-shape partitions, we have the following

Proposition

The above Conjecture holds when partitions λ and μ are both one column. Actually, we have the following stronger inequalities

$$\bar{g}^{\nu}_{(1^j)(1^k)} \geq \bar{g}^{\nu}_{(1^i)(1^l)}, \quad \text{for all partition } \nu,$$

whenever $i < j \leq k < l$ with $j + k = i + l$.

References

[1] S. V. Sam and A. Snowden. Stability patterns in representation theory. *Forum of Mathematics, Sigma*, 3, 2015.

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